Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces

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Abstract

We define Floer homology for a time-independent, or autonomous Hamiltonian on a symplectic manifold with contact type boundary, under the assumption that its 1-periodic orbits are transversally nondegenerate. Our construction is based on Morse-Bott techniques for Floer trajectories. Our main motivation is to understand the relationship between linearized contact homology of a fillable contact manifold and symplectic homology of its filling.

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Contents

1 Introduction

One crucial hypothesis in the definition of Floer homology [12] of a Hamiltonian H on a symplectic manifold (W, ω) is that the 1-periodic orbits of the Hamiltonian vector field X_H are nondegenerate. Unless they are all constant – which happens if the Hamiltonian is C^2-small – this forces H to be time-dependent. The purpose of this paper is to define Floer homology for a time-independent, or autonomous Hamiltonian $H : W \to \mathbb{R}$ under the assumption that its 1-periodic orbits are transversally nondegenerate. This last condition is generic in the space of autonomous Hamiltonians.

Although this generalization of Floer homology is interesting by itself, our main motivation is to understand the relationship between linearized contact homology of a fillable contact manifold (M, ξ) and symplectic homology of the filling (W, ω). In this case there is a natural class of time-independent Hamiltonians on W whose nonconstant 1-periodic orbits correspond precisely to closed Reeb orbits on $M = \partial W$, and for which the Floer trajectories can be related to holomorphic cylinders in the symplectization $M \times \mathbb{R}$ [3]. The goal of the present paper is to relate the Floer trajectories of a specific time-dependent perturbation to the Floer trajectories of the unperturbed Hamiltonian. Thus Floer homology for time-independent Hamiltonians serves as a bridge between symplectic homology and linearized contact homology. Moreover, the moduli spaces of Floer trajectories for autonomous Hamiltonians are related to the moduli spaces defining S^1-equivariant symplectic homology [4, 3].

The Morse-Bott analysis in this paper is, to the best of our knowledge, new to the literature, being based on ideas contained in the first author's Ph.D. dissertation [1] within the context of contact homology. Although our situation is that of critical manifolds of dimension one, the complexity of the analytical setup is the same as that of the higher dimensional case.

We must mention at this point Frauenfelder's inspired approach [16, Appendix A] in which he defines a complex for a Morse-Bott function on a finite dimensional manifold via "flow lines with cascades" – these being our Floer trajectories with gradient fragments – and in which, without proving the correspondence with gradient trajectories for some perturbed Morse function, he directly shows deformation invariance of the resulting chain complex.

We now describe the structure of the paper. We give in the introduction only a loose statement of our main Correspondence Theorem 3.7 and we recall in Section 2 the construction of symplectic homology. Although this is well-known to specialists we still need to establish notations, and we seize the occasion to set up a general framework using the Novikov ring and nontrivial homotopy classes of periodic orbits.

Section 3 describes the Morse-Bott complex and formally states the Correspondence Theorem 3.7. The latter is complemented by Proposition 3.9 which describes how the coherent orientation signs for the Morse-Bott complex are related to the ones for the Floer complex.

Section 4 contains the proofs of the previous transversality, compactness,

gluing and orientation statements. Finally, the Appendix contains the statements concerned with the asymptotic behaviour of the various types of Floer trajectories that we use. These asymptotic estimates enter crucially in the proof of the compactness statements, as well as in the definition of the Fredholm setup for gluing.

We end the introduction with an informal presentation of our results. Let $H : W \to \mathbb{R}$ be an autonomous Hamiltonian defined on a symplectic manifold (W, ω). We assume that H is a Morse function and that the nonconstant 1-periodic orbits of H are transversally nondegenerate. The set $\mathcal{P}(H)$ of 1-periodic orbits of H is the set of critical points of the Hamiltonian action functional and consists of isolated elements $\gamma_{\widetilde{p}}$ corresponding to critical points $\widetilde{p} \in \mathrm{Crit}(H)$, and of nonisolated elements coming in families S_γ which are Morse-Bott nondegenerate circles. These correspond to reparametrizations of some given orbit $\gamma \in \mathcal{P}(H)$, with $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to W$.

For each circle S_γ we choose a perfect Morse function $f_\gamma : S_\gamma \to \mathbb{R}$ with exactly one maximum *Max* and one minimum *min*. We denote by γ_{min}, γ_{Max} the orbits in S_γ corresponding to the minimum and the maximum of f_γ respectively. We choose a chart $S^1 \times \mathbb{R}^{2n-1} \ni (\tau, p)$ and a smooth cut-off function $\rho_\gamma : S^1 \times \mathbb{R}^{2n-1} \to \mathbb{R}$ in the neighbourhood of each $\gamma(S^1) \subset W$, and we denote by $\ell_\gamma \in \mathbb{Z}^+$ the maximal positive integer such that $\gamma(\theta + 1/\ell_\gamma) = \gamma(\theta)$, $\theta \in S^1$.

Following [8], for $\delta > 0$ small enough the time-dependent Hamiltonian

$$H_\delta : S^1 \times W \to \mathbb{R},$$

$$H_\delta(\theta, \tau, p) := H - \delta \sum_{S_\gamma} \rho_\gamma(\tau, p) f_\gamma(\tau - \ell_\gamma \theta)$$

has only nondegenerate 1-periodic orbits. Moreover, these are of the following two types: they are either constant orbits $\gamma_{\widetilde{p}}$ corresponding to critical points $\widetilde{p} \in \mathrm{Crit}(H)$, or they are nonconstant orbits of the form $\gamma_p \in \mathcal{P}(H)$ for $p \in \mathrm{Crit}(f_\gamma)$. Thus, out of each circle S_γ of periodic orbits for H there are exactly two orbits surviving for H_δ, namely γ_{min} and γ_{Max}.

Let J be a generic time-dependent almost complex structure on W. Given $p \in \mathrm{Crit}(f_{\overline{\gamma}})$, $q \in \mathrm{Crit}(f_{\underline{\gamma}})$ we denote by

$$\mathcal{M}(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$$

the moduli space of Floer trajectories for the pair (H_δ, J) modulo reparametrization, with negative asymptote $\overline{\gamma}_p$ and positive asymptote $\underline{\gamma}_q$. We also denote by

$$\mathcal{M}(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$$

the moduli space of Floer trajectories for the pair (H, J) modulo reparametrization, with negative asymptote in $S_{\overline{\gamma}}$ and positive asymptote in $S_{\underline{\gamma}}$. Our goal is to describe the moduli spaces of the first type in terms of moduli spaces of the second type.

We denote by

$$\mathcal{M}(p, q; H, \{f_\gamma\}, J)$$

the moduli space of Floer trajectories for the pair (H, J) with intermediate gradient fragments, consisting of tuples

$$[\mathbf{u}] = (c_m, [u_m], c_{m-1}, [u_{m-1}], \ldots, [u_1], c_0), \quad m \geq 0$$

such that:

(i) $[u_i] \in \mathcal{M}(S_{\gamma_i}, S_{\gamma_{i-1}}; H, J)$, $i = 1, \ldots, m$ with $\gamma_m := \overline{\gamma}$, $\gamma_0 := \underline{\gamma}$;

(ii) c_m is a semi-infinite gradient trajectory of $f_{\overline{\gamma}} = f_{\gamma_m}$ connecting $\overline{\gamma}_p$ to the endpoint of u_m;

(iii) c_j, $j = 1, \ldots, m-1$ is a finite gradient trajectory of f_{γ_j} connecting the endpoints of u_{j+1} and u_j;

(iv) c_0 is a semi-infinite gradient trajectory of $f_{\underline{\gamma}} = f_{\gamma_0}$ connecting the endpoint of u_1 to $\underline{\gamma}_q$.

We give a pictogram of such an element $[\mathbf{u}]$ with $m \geq 1$ in Figure 4 on page 45, where one should read c_i instead of v_i. If $m = 0$ such an element $[\mathbf{u}]$ is simply an infinite gradient trajectory of some f_γ. Let us note that, just as the space of Floer trajectories for a nondegenerate Hamiltonian can be compactified by adding "broken" Floer trajectories, the space of Floer trajectories with intermediate gradient fragments can be compactified by adding "broken" such objects, with an obvious meaning. We denote by $\overline{\mathcal{M}}(p, q; H, \{f_\gamma\}, J)$ these compactified moduli spaces.

Our main result is the following comparison theorem.

Theorem. *The following assertions hold.*

(i) any sequence $[v_n] \in \mathcal{M}(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$, $\delta_n \to 0$ *converges to an element of* $\overline{\mathcal{M}}(p, q; H, \{f_\gamma\}, J)$*;*

(ii) any element of $\overline{\mathcal{M}}(p, q; H, \{f_\gamma\}, J)$ *can be obtained as such a limit;*

(iii) there is a bijective correspondence between elements of $\mathcal{M}(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ *and elements of* $\mathcal{M}(p, q; H, \{f_\gamma\}, J)$ *if the difference of index of the endpoints is equal to one, or equivalently if the moduli spaces have dimension zero.*

The rigorous forms for the statements (i), (ii), (iii) are given in Proposition 4.7, Proposition 4.22 and Theorem 3.7 respectively. Unsurprisingly, the Fredholm setup for the previous theorem uses Sobolev norms with exponential weights since we have degenerate asymptotics. Similarly, due to the convergence estimates in the Appendix, there are such weights centered on the portions of the Floer cylinders approaching gradient fragments. For each peak in the weight, there is a special section supported around this peak which has constant norm

with respect to $\delta \to 0$. For each gradient fragment this section corresponds to the reparametrization shift of the underlying gradient trajectory. As $\delta \to 0$, the corresponding peak explodes and thus forbids all infinitesimal variations except for the single degree of freedom coming from Morse theory.

To be useful for homological calculations the above theorem needs to be complemented by a statement concerning signs. We describe in Section 4.4 how to construct coherent orientations on the relevant spaces of Fredholm operators and how to obtain signs $\epsilon(\mathbf{u})$ and $\epsilon(u_\delta)$ for elements $[\mathbf{u}] \in \mathcal{M}(p,q;H,\{f_\gamma\},J)$ and $u_\delta \in \mathcal{M}(\overline{\gamma}_p,\underline{\gamma}_q;H_\delta,J)$ when the corresponding moduli spaces are zero-dimensional. We recall in Remark 3.2 the definition of *good orbits* borrowed from Symplectic Field Theory, where it plays a crucial role in all orientation and signs problems. In the following statement we denote again by $m \geq 0$ the number of nonconstant Floer trajectories involved in $\mathbf{u}$.

Proposition 3.9. *Assume the moduli spaces under consideration have dimension zero. The bijective correspondence between elements $u_\delta \in \mathcal{M}(\overline{\gamma}_p,\underline{\gamma}_q;H_\delta,J)$ and $[\mathbf{u}] \in \mathcal{M}(p,q;H,\{f_\gamma\},J)$ changes signs as follows:*

(i) If $m \geq 1$ we have
$$\epsilon(\mathbf{u}) = (-1)^{m-1}\epsilon(u_\delta);$$

(ii) If $m = 0$ we have $\mathbf{u} = u_\delta$ and $\epsilon(\mathbf{u}) = \epsilon(u_\delta)$, p is the minimum and q is the maximum of the same function f_γ, the moduli space $\mathcal{M}(p,q;H,\{f_\gamma\},J)$ consists of the two gradient lines of f_γ running from p to q, and their signs are different if and only if the underlying orbit γ is good.

This result has two pleasant consequences. On the one hand we can construct a "Morse-Bott" chain complex which computes symplectic homology by counting with suitable signs rigid elements in the moduli spaces $\mathcal{M}(p,q;H,\{f_\gamma\},J)$. On the other hand, this chain complex singles out algebraically the good orbits and can be used to relate the symplectic homology of a manifold (W,ω) with contact type boundary to the linearized contact homology of its boundary – the latter being defined by a chain complex involving only good orbits. As already mentioned at the beginning of this section, this is achieved in [3].

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2 Symplectic homology

We define in this section the symplectic homology groups of a symplectically aspherical manifold with contact type boundary. Our construction is modelled on those of Cieliebak, Floer, Hofer and Viterbo [6, 7, 13, 28]. We consider nontrivial homotopy classes of loops and we use the Novikov ring.

Let (W, ω) be a compact symplectic manifold with contact type boundary $M := \partial W$. This means that there exists a vector field X defined in a neighbourhood of M, transverse and pointing outwards along M, and such that

$$\mathcal{L}_X \omega = \omega.$$

Such an X is called a **Liouville vector field**. The 1-form $\lambda := (\iota_X \omega)|_M$ is a contact form on M. We denote by ξ the contact distribution defined by λ. The **Reeb vector field** R_λ is uniquely defined by the conditions $\ker \omega|_M = \langle R_\lambda \rangle$ and $\lambda(R_\lambda) = 1$. We denote by ϕ_λ the flow of R_λ. The **action spectrum** of (M, λ) is defined by

$$\mathrm{Spec}(M, \lambda) := \{T \in \mathbb{R}^+ \mid \text{ there is a closed } R_\lambda\text{-orbit of period } T\}.$$

We assume throughout this paper the condition

$$\int_{T^2} f^*\omega = 0 \quad \text{for all smooth } f : T^2 \to W. \tag{1}$$

This guarantees that the energy of a Floer trajectory does not depend on its homology class, but only on its endpoints (see below). Condition (1) plays an important role in the Morse-Bott description of the symplectic homology groups. Our main class of examples is provided by exact symplectic forms.

Let ϕ be the flow of X. We parametrize a neighbourhood U of M by

$$G : M \times [-\delta, 0] \to U, \qquad (p, t) \mapsto \phi^t(p).$$

Then $d(e^t\lambda)$ is a symplectic form on $M \times \mathbb{R}^+$ and G satisfies $G^*\omega = d(e^t\lambda)$. We denote

$$\widehat{W} := W \bigcup_G M \times \mathbb{R}^+$$

and endow it with the symplectic form

$$\widehat{\omega} := \begin{cases} \omega, & \text{on } W, \\ d(e^t\lambda), & \text{on } M \times \mathbb{R}^+. \end{cases}$$

Given a time-dependent Hamiltonian $H : S^1 \times \widehat{W} \to \mathbb{R}$, we define the **Hamiltonian vector field** X_H^θ by

$$\widehat{\omega}(X_H^\theta, \cdot) = dH_\theta, \qquad \theta \in S^1 = \mathbb{R}/\mathbb{Z},$$

where $H_\theta := H(\theta, \cdot)$. We denote by ϕ_H the flow of X_H^θ, defined by $\phi_H^0 = \mathrm{Id}$ and

$$\frac{d}{d\theta}\phi_H^\theta(x) = X_H^\theta(\phi_H^\theta(x)), \qquad \theta \in \mathbb{R}.$$

Let $\mathcal{H}$ be the set of **admissible Hamiltonians**, consisting of functions $H : S^1 \times \widehat{W} \to \mathbb{R}$ which satisfy

(i) $H < 0$ on W;

(ii) $H(\theta, p, t) = \alpha e^t + \beta$ for t large enough, with $\alpha \notin \mathrm{Spec}(M, \lambda)$;

(iii) every 1-periodic orbit $\gamma : S^1 \to \widehat{W}$ of X_H^θ is nondegenerate, i.e.

$$\det\left(\mathbb{1} - d\phi_H^1(\gamma(0))\right) \neq 0.$$

We denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of X_H^θ and by $\mathcal{P}^a(H)$ the set of 1-periodic orbits in a given free homotopy class a in $\widehat{W}$.

Let $\mathcal{J}$ denote the set of **admissible almost complex structures**

$$J : S^1 \to \mathrm{End}(T\widehat{W}), \qquad J^2 = -\mathbb{1}$$

which are compatible with $\widehat{\omega}$ and have the following standard form for t large enough:

$$\begin{cases} J_{(p,t)}|_\xi &= J_0, \\ J_{(p,t)}\frac{\partial}{\partial t} &= R_\lambda. \end{cases} \tag{2}$$

Here J_0 is any compatible complex structure on the symplectic bundle $(\xi, d\lambda)$ which is independent of θ and t.

Let us fix a reference loop $l_a : S^1 \to \widehat{W}$ for each free homotopy class a in $\widehat{W}$ such that $[l_a] = a$. If a is the trivial homotopy class we choose l_a to be a constant loop. Recall that free homotopy classes of loops in $\widehat{W}$ are in one-to-one correspondence with conjugacy classes in $\pi_1(\widehat{W})$. As a consequence, the inverse a^{-1} of a free homotopy class is well-defined. We require that $l_{a^{-1}}$ coincides with the loop l_a with the opposite orientation.

The **Hamiltonian action functional** acts on pairs $(\gamma, [\sigma])$ consisting of a loop $\gamma \in C^\infty(S^1, \widehat{W})$ and the homology class (rel boundary) of a map $\sigma : \Sigma \to \widehat{W}$ defined on a Riemann surface Σ with two boundary components $\partial_0\Sigma$ (with the opposite boundary orientation) and $\partial_1\Sigma$ (with the boundary orientation), which satisfies

$$\sigma|_{\partial_0\Sigma} = l_{[\gamma]}, \qquad \sigma|_{\partial_1\Sigma} = \gamma. \tag{3}$$

Its values are defined by

$$\mathcal{A}_H(\gamma, [\sigma]) := -\int_\Sigma \sigma^*\widehat{\omega} - \int_{S^1} H(\theta, \gamma(\theta))\, d\theta. \tag{4}$$

The differential $d\mathcal{A}_H(\gamma, [\sigma]) : C^\infty(S^1, \gamma^*T\widehat{W}) \to \mathbb{R}$ is given by

$$d\mathcal{A}_H(\gamma, [\sigma])\zeta := \int_{S^1} \widehat{\omega}(\dot\gamma - X_H^\theta(\gamma), \zeta)\, d\theta.$$

Therefore the critical points of $\mathcal{A}_H$ are pairs $(\gamma, [\sigma])$ such that $\gamma \in \mathcal{P}(H)$. We fix from now on, for each $\gamma \in \mathcal{P}(H)$, a map σ_γ satisfying (3); then the set of all pairs $(\gamma, [\sigma])$ can be identified with $H_2(W; \mathbb{Z})$ for fixed γ.

Let us choose a symplectic trivialization

$$\Phi_a : S^1 \times \mathbb{R}^{2n} \to l_a^* T\widehat{W}$$

for each free homotopy class a in $\widehat{W}$. If a is the trivial homotopy class we choose the trivialization to be constant. Moreover, we require that $\Phi_{a^{-1}}(\theta, \cdot) = \Phi_a(-\theta, \cdot)$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. For each $\gamma \in \mathcal{P}(H)$ there exists a unique (up to homotopy) trivialization

$$\Phi_\gamma : \Sigma \times \mathbb{R}^{2n} \to \sigma_\gamma^* T\widehat{W}$$

such that $\Phi_\gamma = \Phi_{[\gamma]}$ on $\partial_0 \Sigma \times \mathbb{R}^{2n}$. Let

$$\Psi : [0,1] \to \mathrm{Sp}(2n), \qquad \Psi(\theta) := \Phi_\gamma^{-1} \circ d\phi_H^\theta(\gamma(0)) \circ \Phi_\gamma. \tag{5}$$

Because γ is nondegenerate we can define the **Conley-Zehnder index** $\mu(\gamma)$ by

$$\mu(\gamma) := \mu(\gamma, \sigma_\gamma) := -\mu_{CZ}(\Psi), \tag{6}$$

where $\mu_{CZ}(\Psi)$ is the Conley-Zehnder index of a path of symplectic matrices [23].

Remark 2.1. If, in the previous construction, we replace σ_γ with $\sigma_\gamma \# A$ for some $A \in H_2(W;\mathbb{Z})$, then the resulting index will be

$$\mu(\gamma, \sigma_\gamma \# A) = \mu(\gamma, \sigma_\gamma) - 2\langle c_1(TW), A\rangle. \tag{7}$$

We define the **Novikov ring** Λ_ω as the set of formal linear combinations $\lambda := \sum_{A \in H_2(W;\mathbb{Z})} \lambda_A e^A$, $\lambda_A \in \mathbb{Z}$ such that

$$\#\{A \mid \lambda_A \neq 0,\ \omega(A) \leq c\} < \infty$$

for all $c > 0$. The multiplication in Λ_ω is given by

$$\lambda * \lambda' := \sum_{A,B \in H_2(W;Z)} \lambda_A \lambda'_B e^{A+B}.$$

We note that, if ω is exact, then $\Lambda_\omega = \mathbb{Z}[H_2(W;\mathbb{Z})]$. We define a grading on Λ_ω by $|e^A| := -2\langle c_1(TW), A\rangle$. For each free homotopy class a in $\widehat{W}$ and each admissible Hamiltonian H we define the **symplectic chain group** $SC_*^a(H)$ as the free Λ_ω-module generated by elements $\gamma \in \mathcal{P}^a(H)$. The grading is given by

$$|e^A \gamma| := \mu(\gamma) - 2\langle c_1(TW), A\rangle.$$

We define the space of Floer trajectories $\widehat{\mathcal{M}}^A(\overline{\gamma}, \underline{\gamma}; H, J)$ as the set of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ of the equation

$$\partial_s u + J_\theta(\partial_\theta u - X_H^\theta) = 0, \tag{8}$$

such that

$$\lim_{s\to-\infty} u(s,\theta) = \overline{\gamma}(\theta), \qquad \lim_{s\to\infty} u(s,\theta) = \underline{\gamma}(\theta), \qquad \lim_{s\to\pm\infty} \partial_s u = 0 \tag{9}$$

uniformly in θ and

$$[\sigma_{\overline{\gamma}} \# u] = [\sigma_{\underline{\gamma}} \# A]. \tag{10}$$

Remark 2.2. Under the nondegeneracy assumption on $\overline{\gamma}$, $\underline{\gamma}$ condition (9) is equivalent to the finiteness of the energy

$$\mathcal{E}(u) := \mathcal{E}_{J,H}(u) := \frac{1}{2}\int_{\mathbb{R}\times S^1} \left(|\partial_s u|_\theta^2 + |\partial_t u - X_H^\theta|_\theta^2\right)\, ds d\theta. \tag{11}$$

Because $\overline{\gamma}$, $\underline{\gamma}$ are nondegenerate the linearized operator $D_u : W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}) \to L^p(\mathbb{R}\times S^1, u^*T\widehat{W})$, $p>2$ given by

$$D_u\zeta := \nabla_s\zeta + J_\theta\nabla_\theta\zeta + (\nabla_\zeta J_\theta)\partial_\theta u - \nabla_\zeta\left(J_\theta X_H^\theta\right),\ u\in\widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J) \tag{12}$$

is Fredholm with index

$$\mathrm{ind}(D_u) = \mu(\overline{\gamma}) - \mu(\underline{\gamma}) + 2\langle c_1(TW), A\rangle. \tag{13}$$

An almost complex structure $J\in\mathcal{J}$ is called **regular for** $u\in\widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)$ if D_u is surjective, and it is called **regular** if D_u is surjective for all $\overline{\gamma},\underline{\gamma}\in\mathcal{P}(H)$, $A\in H_2(W;\mathbb{Z})$ and $u\in\widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)$. It is proved in [15] that the space $\mathcal{J}_{\mathrm{reg}}(H)$ of regular almost complex structures is of the second category in $\mathcal{J}$. For every $J\in\mathcal{J}_{\mathrm{reg}}(H)$ the space $\widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)$ is a smooth manifold of dimension $\mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle$. From now on we fix some $J\in\mathcal{J}_{\mathrm{reg}}(H)$.

If $\overline{\gamma}\neq\underline{\gamma}$ or $A\neq 0$, the additive group $\mathbb{R}$ acts freely on $\widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)$ by $s_0\cdot u(\cdot,\cdot) := u(s_0+\cdot,\cdot)$. We define the **moduli space of Floer trajectories** by

$$\mathcal{M}^A(\overline{\gamma},\underline{\gamma};H,J) := \widehat{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)/\mathbb{R}.$$

Its dimension is

$$\dim\,\mathcal{M}^A(\overline{\gamma},\underline{\gamma};H,J) := \mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle - 1.$$

If $\overline{\gamma}=\underline{\gamma}$ and $A=0$, the space $\widehat{\mathcal{M}}^0(\overline{\gamma},\overline{\gamma};H,J)$ consists of a single point, correponding to a constant solution (i.e. independent of s). The $\mathbb{R}$ action is then trivial and we define the moduli space by $\mathcal{M}^0(\overline{\gamma},\overline{\gamma};H,J) := \widehat{\mathcal{M}}^0(\overline{\gamma},\overline{\gamma};H,J)$. A straightforward application of the maximum principle [28] using the special form of admissible Hamiltonians for large t shows that all solutions of equations (8) and (9) are contained in a compact set. Moreover, by condition (1), there are no J-holomorphic spheres that can bubble off. Therefore the moduli space $\mathcal{M}^A(\overline{\gamma},\underline{\gamma};H,J)$ can be compactified [12] to a space $\overline{\mathcal{M}}^A(\overline{\gamma},\underline{\gamma};H,J)$ consisting of all tuples

$$([u_k],[u_{k-1}],\ldots,[u_1]),\qquad [u_i]\in\mathcal{M}^{A_i}(\overline{\gamma}_i,\underline{\gamma}_i;H,J)$$

such that $\underline{\gamma}_1=\underline{\gamma}$, $\overline{\gamma}_i=\underline{\gamma}_{i+1}$, $\overline{\gamma}_k=\overline{\gamma}$ and $\sum_i A_i = A$. We call such a tuple $([u_k],[u_{k-1}],\ldots,[u_1])$ a **broken trajectory** of level k. The topology of the compactified moduli space is described by the following notion of convergence: a sequence $[u^\nu]\in\mathcal{M}^A(\overline{\gamma},\underline{\gamma};H,J)$ is said to converge to the broken trajectory

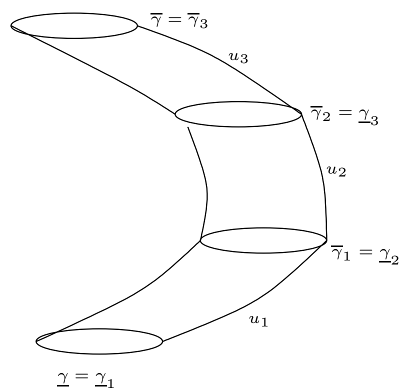

Figure 1: Broken trajectory.

$([u_k], [u_{k-1}], \ldots, [u_1])$ if there exist sequences $s_i^\nu \in \mathbb{R}$, $1 \le i \le k$ such that $s_i^\nu \cdot u^\nu$ converges uniformly on compact sets to u_i.

If the space $\widehat{\mathcal{M}}^A(\overline{\gamma}, \underline{\gamma}; H, J)$ is nonempty then its dimension is strictly positive due to the action of $\mathbb{R}$. In this case, the broken trajectories involved in the compactification have level at most $\dim \widehat{\mathcal{M}}^A(\overline{\gamma}, \underline{\gamma}; H, J)$. In particular, when $\mu(\overline{\gamma}) - \mu(\underline{\gamma}) + 2\langle c_1(TW), A\rangle = 1$ the moduli space $\mathcal{M}^A(\overline{\gamma}, \underline{\gamma}; H, J)$ is compact and consists of a finite number of points. In this situation one can associate a sign $\epsilon(u)$ to each element $[u]$ of this moduli space [13] (see also Section 4.4). We define the **Floer differential**

$$\partial : SC_*^a(H) \to SC_{*-1}^a(H)$$

by

$$\partial \overline{\gamma} := \sum_{\substack{\underline{\gamma},\ A \\ \mu(\overline{\gamma}) - \mu(\underline{\gamma}) + 2\langle c_1(TW), A\rangle = 1}} \sum_{[u] \in \mathcal{M}^A(\overline{\gamma}, \underline{\gamma}; H, J)} \epsilon(u) e^A \underline{\gamma}. \tag{14}$$

According to Floer [12] we have $\partial^2 = 0$. We define the **symplectic homology groups** of the pair (H, J) by

$$SH_*^a(H, J) := H_*(SC_*^a(H), \partial).$$

Remark 2.3. In view of condition (1) the Novikov ring Λ_ω can be replaced by $\mathbb{Z}[H_2(W; \mathbb{Z})]$, or even by $\mathbb{Z}$ at the price of losing the grading. Indeed, the energy of a Floer trajectory depends only on its endpoints, hence the moduli spaces $\overline{\mathcal{M}}(\overline{\gamma}, \underline{\gamma}; H, J) := \bigcup_A \overline{\mathcal{M}}^A(\overline{\gamma}, \underline{\gamma}; H, J)$ are compact. Therefore the sum (14) involves only a finite number of classes A.

By a standard argument [12] the groups $SH_*^a(H, J)$ do not depend on $J \in \mathcal{J}_{\mathrm{reg}}(H)$. Nevertheless, they *do* depend on H and, in order to obtain an invariant of (W, ω), we need an additional algebraic limit construction. We define an

admissible homotopy of Hamiltonians as a map $H : \mathbb{R} \times S^1 \times \widehat{W} \to \mathbb{R}$ with the following properties:

(i) $H(s, \cdot, \cdot) = H_- \in \mathcal{H}$ for $s \le -1$, $H(s, \cdot, \cdot) = H_+ \in \mathcal{H}$ for $s \ge 1$;

(ii) $H < 0$ on W and there exist $t_0 \ge 0$ and functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ such that, for all $t \ge t_0$, we have

$$H(s, t, p) = \alpha(s)e^t + \beta(s);$$

(iii) $\partial_s H \ge 0$.

An **admissible homotopy of almost complex structures** is a map $J : \mathbb{R} \to \mathcal{J}$ such that $J(s) = J_-$ for $s \le -1$ and $J(s) = J_+$ for $s \ge 1$. Given an admissible homotopy of Hamiltonians one defines regular admissible homotopies of almost complex structures in the usual way, by linearizing the equation

$$\partial_s u(s, \theta) + J(s, \theta, u(s, \theta))(\partial_\theta u(s, \theta) - X_H^\theta(s, \theta, u(s, \theta))) = 0, \tag{15}$$

subject to the limit conditions

$$\lim_{s \to -\infty} u(s, \cdot) = \overline{\gamma} \in \mathcal{P}(H_-), \qquad \lim_{s \to \infty} u(s, \cdot) = \underline{\gamma} \in \mathcal{P}(H_+). \tag{16}$$

Regular admissible homotopies of almost complex structures form again a set of the second category in the space of admissible homotopies and the rigid behaviour of H for $t \ge t_0$, together with the condition $\partial_s H \ge 0$, ensures again that solutions of (15) and (16) stay in a compact set (see [22]). The usual count of solutions of (15) and (16) induces the **monotonicity morphism**

$$\sigma : SH_*^a(H_-) \to SH_*^a(H_+), \tag{17}$$

which does not depend on the choice of admissible homotopy connecting H_- and H_+. These morphisms form a direct system on the set $\{SH_*^a(H),\ H \in \mathcal{H}\}$ and we define the **symplectic homology groups** of (W, ω) by

$$SH_*^a(W, \omega) := \varinjlim_{H \in \mathcal{H}} SH_*^a(H).$$

According to [6, Lemma 3.7] and [28, Theorem 1.7] these groups do not depend on the choice of the Liouville vector field X.

3 The Morse-Bott chain complex

In this section we apply the Morse-Bott formalism of [1] to the case of Hamiltonians $H : \widehat{W} \to \mathbb{R}$ having circles of 1-periodic orbits.

We denote by $\mathcal{P}_\lambda$ the set of closed unparametrized R_λ-orbits in M. For each free homotopy class of loops b in M we denote by $\mathcal{P}_\lambda^b$ the set of all $\gamma \in \mathcal{P}_\lambda$ in the homotopy class b. The inclusion $i : M \hookrightarrow W$ induces a map (still denoted by

i) between the sets of free homotopy classes of loops in M and W respectively. For each free homotopy class a in W we denote

$$\mathcal{P}_\lambda^{i^{-1}(a)} := \bigcup_{b\in i^{-1}(a)} \mathcal{P}_\lambda^b.$$

We assume in this section that the closed Reeb orbits on M are transversally nondegenerate in M. This means that, for every orbit γ of period $T>0$, we have

$$\det\left(\mathbb{1} - d\phi_\lambda^T(\gamma(0))|_\xi\right) \neq 0.$$

This can always be achieved by an arbitrarily small perturbation of λ or, equivalently, of X, and such perturbations do not change the symplectic homology groups. If all orbits $\gamma\in\mathcal{P}_\lambda$ are transversally nondegenerate one can assign to each of them a Conley-Zehnder index $\mu_{CZ}(\gamma)$ according to the following recipe.

We fix a reference loop $l_b : S^1 \to M$ for each free homotopy class b in M such that $[l_b]=b$. If b is the trivial homotopy class we choose l_b to be a constant loop and we require that $l_{b^{-1}}$ coincides with l_b with the opposite orientation. We define symplectic trivializations

$$\Phi_b : S^1\times\mathbb{R}^{2n-2} \to l_b^*\xi$$

as follows. For each class b we choose a homotopy $h_{ab} : S^1\times[0,1]\to W$ from l_a, $a=i(b)$ to l_b such that

$$h_{a^{-1}b^{-1}}(\tau,\cdot) = h_{ab}(-\tau,\cdot). \tag{18}$$

We extend the trivialization $\Phi_a : S^1\times\mathbb{R}^{2n}\to l_a^*T\widehat{W}$ over the homotopy h_{ab} to get a trivialization $\Phi_b' : S^1\times\mathbb{R}^{2n}\to l_b^*T\widehat{W}$. This trivialization is homotopic to another one, still denoted Φ_b', such that

$$\begin{aligned} \Phi_b'(S^1\times\mathbb{R}^{2n-2}\times\{0\}\times\{0\}) &= l_b^*\xi, \\ \Phi_b'(S^1\times\{0\}\times\mathbb{R}\times\{0\}) &= l_b^*\langle\frac{\partial}{\partial t}\rangle, \\ \Phi_b'(S^1\times\{0\}\times\{0\}\times\mathbb{R}) &= l_b^*\langle R_\lambda\rangle. \end{aligned} \tag{19}$$

We define $\Phi_b := \Phi_b'|_{S^1\times\mathbb{R}^{2n-2}\times\{0\}\times\{0\}}$. If b is the trivial homotopy class we choose h_{ab} to be a path of constant loops, so that Φ_b is constant.

We fix for each $\gamma\in\mathcal{P}_\lambda$ a map $\sigma_\gamma : \Sigma\to M$, where Σ is a Riemann surface with two boundary components $\partial_0\Sigma$ (with the opposite boundary orientation) and $\partial_1\Sigma$ (with the boundary orientation), satisfying

$$\sigma|_{\partial_0\Sigma} = l_{[\gamma]}, \qquad \sigma|_{\partial_1\Sigma}=\gamma. \tag{20}$$

For each $\gamma\in\mathcal{P}_\lambda$ there exists a unique (up to homotopy) trivialization

$$\Phi_\gamma : \Sigma\times\mathbb{R}^{2n-2}\to\sigma_\gamma^*\xi$$

such that $\Phi_\gamma = \Phi_{[\gamma]}$ on $\partial_0\Sigma \times \mathbb{R}^{2n-2}$. Let

$$\Psi : [0,T] \to \mathrm{Sp}(2n-2), \quad \Psi(\tau) := \Phi_\gamma^{-1} \circ d\phi_\lambda^\tau(p) \circ \Phi_\gamma, \quad p \in \mathrm{im}\,\gamma. \tag{21}$$

Because γ is nondegenerate we can define the **Conley-Zehnder index** $\mu(\gamma)$ by

$$\mu(\gamma) := \mu(\gamma, \sigma_\gamma) := \mu_{CZ}(\Psi), \tag{22}$$

where $\mu_{CZ}(\Psi)$ is the Conley-Zehnder index of a path of symplectic matrices [23].

Remark 3.1. If, in the previous construction, we replace σ_γ with $\sigma_\gamma \# A$ for some $A \in H_2(M;\mathbb{Z})$, then the resulting index will be

$$\mu(\gamma, \sigma_\gamma \# A) = \mu(\gamma, \sigma_\gamma) + 2\langle c_1(\xi), A\rangle. \tag{23}$$

Note that $c_1(\xi) = i^* c_1(TW)$ because $i^*TW = \xi \oplus \langle \frac{\partial}{\partial t}, R_\lambda\rangle$. Moreover, the parity of $\mu(\gamma)$ is well-defined independently of the trivialization of ξ along γ.

Remark 3.2. For each simple orbit $\gamma \in \mathcal{P}_\lambda$ we denote by γ^k, $k \in \mathbb{Z}^+$ its positive iterates. The parity of the Conley-Zehnder index of all the odd, respectively even iterates is the same. If these two parities differ we say that all even iterates γ^{2k}, $k \in \mathbb{Z}^+$ are **bad orbits**. It is proved in [27, Lemma 3.2.4] that the even iterates of a simple orbit γ of period T are bad if and only if $d\phi_\lambda^T(p)|_\xi$, $p \in \mathrm{im}\,\gamma$ has an odd number of real negative eigenvalues strictly smaller than -1 (see also Lemma 4.25). The orbits in $\mathcal{P}_\lambda$ which are not bad are called **good orbits**.

We define a new class $\mathcal{H}'$ of admissible Hamiltonians consisting of elements $H : \widehat{W} \to \mathbb{R}$ such that

(i) $H|_W$ is a C^2-small Morse function and $H < 0$ on W;

(ii) $H(p,t) = h(t)$ outside W, where $h(t)$ is a strictly increasing function with $h(t) = \alpha e^t + \beta$, $\alpha, \beta \in \mathbb{R}$, $\alpha \notin \mathrm{Spec}(M,\lambda)$ for t bigger than some t_0, and such that $h'' - h' > 0$ on $[0, t_0[$.

Note that the 1-periodic orbits of X_H in W are constant and nondegenerate by assumption (i). A direct computation shows that

$$X_h(p,t) = -e^{-t} h'(t) R_\lambda. \tag{24}$$

The 1-periodic orbits of X_H fall in two classes:

(1) critical points of H in W;

(2) nonconstant 1-periodic orbits of X_h, located on levels $M \times \{t\}$, $t \in]0, t_0[$, which are in one-to-one correspondence with closed $-R_\lambda$-orbits of period $e^{-t}h'(t)$.

Recall that, for every critical point $\widetilde{p} \in \mathrm{Crit}(H)$, the corresponding constant X_H-orbit $\gamma_{\widetilde{p}}$ has Conley-Zehnder index

$$\mu(\gamma_{\widetilde{p}}) = \mathrm{ind}(\widetilde{p}; -H) - n, \qquad n = \frac{1}{2} \dim W,$$

where $\mathrm{ind}(\widetilde{p}; -H)$ is the Morse index of $\widetilde{p}$ with respect to $-H$ [25, Lemma 7.2].

Let $\alpha := \lim_{t \to \infty} e^{-t} H(p, t)$. We denote by $\mathcal{P}_\lambda^{\leq \alpha}$ the set of all $\gamma \in \mathcal{P}_\lambda$ such that $\int \gamma^* \lambda \leq \alpha$. Because H is independent of θ, every orbit $\gamma \in \mathcal{P}_\lambda^{\leq \alpha}$ gives rise to a whole circle of nonconstant 1-periodic orbits γ_H of X_H. We denote by S_γ the set of such orbits and identify S_γ with its image under the natural embedding $S_\gamma \to \widehat{W}$ given by $\gamma_H \mapsto \gamma_H(0)$. Note that all elements of S_γ differ by a shift in the parametrization, and that the γ_H are noninjective if their minimal period is smaller than 1.

Lemma 3.3. *Let $H \in \mathcal{H}'$. Every nonconstant 1-periodic orbit γ_H of H is transversally nondegenerate in $\widehat{W}$.*

Proof. We have to show that the only eigenvector of $d\phi_H^1(\gamma_H(0))$ corresponding to the eigenvalue 1 is $\dot{\gamma}_H(0)$. To this effect we note that ξ is an invariant space and that

$$d\phi_H^1(\gamma_H(0))|_\xi = \left(d\phi_\lambda^{e^{-t} h'(t)} \right)^{-1} (\gamma_H(0))|_\xi.$$

Because we have assumed that all R_λ-orbits are transversally nondegenerate in M, it follows that $d\phi_H^1(\gamma_H(0))|_\xi$ has no eigenvalue equal to one. On the other hand we have

$$d\phi_H^1(\gamma_H(0)) \frac{\partial}{\partial t} = \frac{\partial}{\partial t} - e^{-t}(h'' - h') R_\lambda.$$

The conclusion follows because $h''(t) - h'(t) > 0$. $\square$

For each $\gamma \in \mathcal{P}_\lambda^{\leq \alpha}$ we choose a Morse function $f_\gamma : S_\gamma \to \mathbb{R}$ with exactly one maximum M and one minimum m. We fix from now on an element $H \in \mathcal{H}'$ and, for each $\gamma \in \mathcal{P}_\lambda$ corresponding to a nonconstant $\gamma_H \in \mathcal{P}(H)$, we denote by $\ell_\gamma \in \mathbb{Z}^+$ the maximal positive integer such that $\gamma_H(\theta + \frac{1}{\ell_\gamma}) = \gamma_H(\theta)$ for all $\theta \in S^1$. We choose a symplectic trivialization $\psi := (\psi_1, \psi_2) : U_\gamma \xrightarrow{\sim} V \subset S^1 \times \mathbb{R}^{2n-1}$ between open neighbourhoods $U_\gamma \subset \widehat{W}$ of $\gamma_H(S^1)$ and V of $S^1 \times \{0\}$, such that $\psi_1(\gamma(\theta)) = \ell_\gamma \theta$. Here $S^1 \times \mathbb{R}^{2n-1}$ is endowed with the symplectic form $\omega_0 := \sum_{i=1}^n dq_i \wedge dp_i$, $q_1 \in S^1$, $(p_1, q_2, p_2, \ldots, q_n, p_n) \in \mathbb{R}^{2n-1}$. Let $\rho : S^1 \times \mathbb{R}^{2n-1} \to [0, 1]$ be a smooth cutoff function supported in a small neighbourhood of $S^1 \times \{0\}$ such that $\rho|_{S^1 \times \{0\}} \equiv 1$. For $\delta > 0$ and $(\theta, p, t) \in S^1 \times U_\gamma$ we define

$$H_\delta(\theta, p, t) := h(t) + \delta \rho(\psi(p, t)) f_\gamma(\psi_1(p, t) - \ell_\gamma \theta). \tag{25}$$

The Hamiltonian H_δ coincides with H outside the open sets $S^1 \times U_\gamma$. This is precisely the perturbation described in [8, Proposition 2.2]. It is shown therein that, for δ sufficiently small, the set $\mathcal{P}(H_\delta)$ consists of the following elements:

(1) constant orbits, which are the same as those of H;

(2) nonconstant orbits, which are nondegenerate and form pairs $(\gamma_{min}, \gamma_{Max})$, where $\gamma \in \mathcal{P}_\lambda^{\leq\alpha}$ and γ_{min}, γ_{Max} coincide with the orbits in S_γ starting at the minimum and the maximum of f_γ respectively.

Lemma 3.4. *The periodic orbits $\gamma_{min}, \gamma_{Max} \in \mathcal{P}(H_\delta)$ satisfy*

$$\mu(\gamma_{min}) = \mu(\gamma) + 1, \qquad \mu(\gamma_{Max}) = \mu(\gamma). \tag{26}$$

Proof. We denote by γ_H the 1-periodic orbit of X_H corresponding to $\gamma \in \mathcal{P}_\lambda^{\leq\alpha}$. We define the Robbin-Salamon index of γ_H by

$$\mu_{RS}(\gamma_H) := \mu(\Psi),$$

where $\Psi : [0,1] \to \mathrm{Sp}(2n)$ is given by (5) and $\mu(\Psi)$ is the Robbin-Salamon index of an arbitrary path of symplectic matrices [23, §4]. It is shown in [8, Proposition 2.2] that

$$-\mu(\gamma_{min}) = \mu_{RS}(\gamma_H) - \frac{1}{2}, \qquad -\mu(\gamma_{Max}) = \mu_{RS}(\gamma_H) + \frac{1}{2}. \tag{27}$$

Note that γ_H has the orientation of $-R_\lambda$. Define $\widetilde{\Psi} : [0,1] \to \mathrm{Sp}(2n)$ by

$$\widetilde{\Psi}(\theta) := \Phi_\gamma^{-1}(-\theta) \circ d\phi_H^{-\theta}(\gamma(0)) \circ \Phi_\gamma(0),$$

where $\Phi_\gamma : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n} \to \gamma_H^* T\widehat{W}$ is the trivialization involved in (5). Then $\mu_{RS}(-\gamma_H) = -\mu_{RS}(\gamma_H) = \mu(\widetilde{\Psi})$.

Let $\mathrm{Sp}^*(2n) \subset \mathrm{Sp}(2n)$ be the set of symplectic matrices with no eigenvalue equal to 1 and recall that we have denoted by a free homotopy classes of loops in W and by b free homotopy classes of loops in M. By our choice (18) and (19) of trivializations of $T\widehat{W}$ over the reference loops l_b, $b \in i^{-1}(a)$ we deduce that the path $\widetilde{\Psi}$ is homotopic with endpoint in $\mathrm{Sp}^*(2n)$ to the path

$$[0,1] \to \mathrm{Sp}(2n) \ : \ \theta \mapsto \Psi_\lambda(T\theta) \oplus \begin{pmatrix} 1 & 0 \\ T\theta & 1 \end{pmatrix}.$$

Here $T := e^{-t}(h''(t) - h'(t))$ and $\Psi_\lambda : [0,T] \to \mathrm{Sp}(2n-2)$ is defined by (21). By the symplectic shear axiom for the Robbin-Salamon index [23, Theorem 4.1] the index of the above path is $\mu(\gamma) + \frac{1}{2}$. As a consequence

$$-\mu_{RS}(\gamma_H) = \mu_{RS}(-\gamma_H) = \mu(\gamma) + \frac{1}{2}.$$

Together with (27) this yields the conclusion of the Lemma. □

Let $p \in \mathrm{Crit}(f_\gamma)$; then $\gamma_p \in \mathcal{P}(H_\delta)$ for all $\delta \in]0, \delta_0]$ if δ_0 is small enough, and Lemma 3.4 says that $\mu(\gamma_p) = \mu(\gamma) + \mathrm{ind}(p; f_\gamma)$. If $\widetilde{p}$ is a critical point of H in

W we denote by $\underline{\gamma}_{\widetilde{p}} \in \mathcal{P}(H)$ the corresponding constant orbit. Our goal is to describe the boundary points as $\delta \to 0$ of

$$\mathcal{M}^A_{]0,\delta_0[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J) := \bigcup_{0<\delta<\delta_0} \{\delta\} \times \mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J), \tag{28}$$

with

$$\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) + 2\langle c_1(TW), A\rangle = 1,$$

where

$$\overline{\gamma}, \underline{\gamma} \in \mathcal{P}^{\leq \alpha}_\lambda,\ p \in \mathrm{Crit}(f_{\overline{\gamma}}),\ q \in \mathrm{Crit}(f_{\underline{\gamma}}),\ A \in H_2(W;\mathbb{Z}),\ J \in \mathcal{J}$$

or

$$\overline{\gamma} \in \mathcal{P}^{\leq \alpha}_\lambda,\ p \in \mathrm{Crit}(f_{\overline{\gamma}}),\ q \in \mathrm{Crit}(H),\ A \in H_2(W;\mathbb{Z}),\ J \in \mathcal{J}.$$

Our description is very similar to that of [1] within the setting of contact homology. We fix $J \in \mathcal{J}$, $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}^{\leq \alpha}_\lambda$ and $\widetilde{q} \in \mathrm{Crit}(H)$. We define two Morse-Bott spaces of **Floer trajectories** $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ and $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$ as follows.

For $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}^{\leq \alpha}_\lambda$ we denote by $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ the set of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ of the Floer equation (8) subject to the asymptotic conditions

$$\lim_{s\to-\infty} u(s,\theta) = \overline{\gamma}_H(\theta), \qquad \lim_{s\to\infty} u(s,\theta) = \underline{\gamma}_H(\theta), \qquad \lim_{s\to\pm\infty} \partial_s u = 0 \tag{29}$$

uniformly in θ, with

$$\overline{\gamma}_H \in S_{\overline{\gamma}}, \qquad \underline{\gamma}_H \in S_{\underline{\gamma}} \tag{30}$$

and

$$[\sigma_{\overline{\gamma}} \# u] = [\sigma_{\underline{\gamma}} \# A].$$

It is implicit in the above definition that the orbits $\overline{\gamma}_H$ and $\underline{\gamma}_H$ may vary for different elements of $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$.

For $\overline{\gamma} \in \mathcal{P}^{\leq \alpha}_\lambda$ and $\widetilde{q} \in \mathrm{Crit}(H)$ we denote by $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$ the set of solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ of the Floer equation (8) subject to the asymptotic conditions

$$\lim_{s\to-\infty} u(s,\theta) = \overline{\gamma}_H(\theta), \qquad \lim_{s\to\infty} u(s,\theta) = \widetilde{q}, \qquad \lim_{s\to\pm\infty} \partial_s u = 0 \tag{31}$$

uniformly in θ, with

$$\overline{\gamma}_H \in S_{\overline{\gamma}} \tag{32}$$

and

$$[\sigma_{\overline{\gamma}} \# u] = A.$$

Again, the orbit $\overline{\gamma}_H$ may vary for different elements of $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$.

If $\overline{\gamma} \neq \underline{\gamma}$ or $A \neq 0$, the additive group $\mathbb{R}$ acts freely on $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ and $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$ by $s_0 \cdot u(\cdot, \cdot) := u(s_0 + \cdot, \cdot)$. We define the **Morse-Bott moduli spaces of Floer trajectories** by

$$\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) := \widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)/\mathbb{R}$$

and

$$\mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J) := \widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)/\mathbb{R}.$$

If $\overline{\gamma} = \underline{\gamma}$ and $A = 0$, the space $\widehat{\mathcal{M}}^0(S_{\overline{\gamma}}, S_{\overline{\gamma}}; H, J)$ is diffeomorphic to $S_{\overline{\gamma}}$, consists of constant cylinders (i.e. independent of s) and the $\mathbb{R}$ action is trivial. In this case, we define the Morse-Bott moduli spaces by $\mathcal{M}^0(S_{\overline{\gamma}}, S_{\overline{\gamma}}; H, J) := \widehat{\mathcal{M}}^0(S_{\overline{\gamma}}, S_{\overline{\gamma}}; H, J)$. We have natural evaluation maps

$$\overline{\mathrm{ev}} : \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) \to S_{\overline{\gamma}}, \qquad \underline{\mathrm{ev}} : \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) \to S_{\underline{\gamma}}$$

and

$$\overline{\mathrm{ev}} : \mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J) \to S_{\overline{\gamma}}$$

defined by

$$\overline{\mathrm{ev}}([u]) := \lim_{s \to -\infty} u(s, \cdot), \qquad \underline{\mathrm{ev}}([u]) := \lim_{s \to \infty} u(s, \cdot).$$

In the statement of the next result we denote by $\mathcal{J}'$ the set of almost complex structures $J \in \mathcal{J}$ which are independent of $\theta \in S^1$.

Proposition 3.5. *(i) Given $H \in \mathcal{H}'$, let $\mathcal{J}'(H) \subset \mathcal{J}'$ be the (nonempty and open) set of almost complex structures J such that, for any $x \in \widehat{W}$ located on a simple* 1*-periodic orbit of X_H, we have*

$$[X_H, JX_H](x) \neq 0 \quad \text{and} \quad [X_H, JX_H](x) \notin \langle X_H, JX_H \rangle. \tag{33}$$

There exists a set of second category $\mathcal{J}'_{\mathrm{reg}}(H) \subset \mathcal{J}'(H)$ consisting of almost complex structures J that are regular for all $u \in \widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ or $u \in \widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$ with $\overline{\gamma}$ or $\underline{\gamma}$ being a simple orbit, and such that $J\xi = \xi$, $J\frac{\partial}{\partial t} = R_\lambda$ outside a fixed open neighbourhood of the nonconstant periodic orbits of X_H.

(ii) Given $H \in \mathcal{H}'$, there exists a set of second category $\mathcal{J}_{\mathrm{reg}}(H) \subset \mathcal{J}$ consisting of regular almost complex structures J which, outside a fixed open neighbourhood of the nonconstant periodic orbits of X_H, are independent of θ and satisfy $J\xi = \xi$, $J\frac{\partial}{\partial t} = R_\lambda$.

In each of the previous cases the relevant moduli spaces $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$, $\mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$ are smooth manifolds of dimension

$$\begin{aligned} \dim \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) &= \mu(\overline{\gamma}) - \mu(\underline{\gamma}) + 2\langle c_1(TW), A \rangle, \\ \dim \mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J) &= \mu(\overline{\gamma}) - \mu(\gamma_{\widetilde{q}}) + 2\langle c_1(TW), A \rangle, \end{aligned}$$

and the evaluation maps $\overline{\mathrm{ev}}, \underline{\mathrm{ev}}$ are smooth.

The proof of this statement is given in Section 4. Unless the contrary is explicitly mentioned, all statements in this section hold both for $J \in \mathcal{J}_{\mathrm{reg}}(H)$ or $J \in \mathcal{J}'_{\mathrm{reg}}(H)$, provided one considers moduli spaces with at least one simple asymptotic orbit in the latter case.

Let now $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and fix for each $\gamma \in \mathcal{P}_\lambda^{\le \alpha}$ a metric on S_γ such that R_λ has length one. Let $\mathcal{F}_{\mathrm{reg}}(H,J)$ be the set of **regular Morse functions**, consisting of families $\{f_\gamma\}$, $\gamma \in \mathcal{P}_\lambda^{\le \alpha}$ of perfect Morse functions $f_\gamma : S_\gamma \to \mathbb{R}$ such that all the maps $\overline{\mathrm{ev}}$ are transverse to the unstable manifolds $W^u(p)$, $p \in \mathrm{Crit}(f_\gamma)$, all the maps $\underline{\mathrm{ev}}$ are transverse to the stable manifolds $W^s(p)$, $p \in \mathrm{Crit}(f_\gamma)$ and all pairs

$$(\overline{\mathrm{ev}}, \underline{\mathrm{ev}}) : \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J) \to S_{\overline{\gamma}} \times S_{\underline{\gamma}},$$

$$(\overline{\mathrm{ev}}, \underline{\mathrm{ev}}) : \mathcal{M}^{A_1}(S_{\overline{\gamma}}, S_{\gamma_1}; H, J) \;{}_{\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}} \mathcal{M}^{A_2}(S_{\gamma_1}, S_{\underline{\gamma}}; H, J) \to S_{\overline{\gamma}} \times S_{\underline{\gamma}} \tag{34}$$

are transverse to products $W^u(p) \times W^s(q)$, $p \in \mathrm{Crit}(f_{\overline{\gamma}})$, $q \in \mathrm{Crit}(f_{\underline{\gamma}})$. Here and in the sequel the unstable and stable manifolds are understood with respect to ∇f_γ. Denote by $C_p^\infty(S_\gamma, \mathbb{R})$ the set of perfect Morse functions on S_γ.

Lemma 3.6. *The set $\mathcal{F}_{\mathrm{reg}}(H,J)$ is of the second Baire category in the space $\prod_{\gamma \in \mathcal{P}_\lambda^{\le\alpha}} C_p^\infty(S_\gamma, \mathbb{R})$.*

Proof. The first two transversality conditions on $\overline{\mathrm{ev}}$, $\underline{\mathrm{ev}}$ are satisfied if and only if the maximum of each function f_γ is a regular value of all the evaluation maps $\overline{\mathrm{ev}}$ having S_γ as target space, and if the minimum of each f_γ is a regular value of all the evaluation maps $\underline{\mathrm{ev}}$ mapping to S_γ. The third transversality condition requires in addition that each pair $(\overline{M}, \underline{m}) \in S_{\overline{\gamma}} \times S_{\underline{\gamma}}$, with $\overline{M}$ the maximum of $f_{\overline{\gamma}}$ and $\underline{m}$ the minimum of $f_{\underline{\gamma}}$, is a regular value of $(\overline{\mathrm{ev}}, \underline{\mathrm{ev}})$.

By Sard's theorem the minimum and maximum of each f_γ can be chosen inside a set of second category in S_γ. The conclusion follows. $\square$

Let now $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and $\{f_\gamma\} \in \mathcal{F}_{\mathrm{reg}}(H,J)$. For $p \in \mathrm{Crit}(f_\gamma)$ we denote the Morse index by

$$\mathrm{ind}(p) := \dim W^u(p; \nabla f_\gamma).$$

Let $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}_\lambda^{\le\alpha}$ and $p \in \mathrm{Crit}(f_{\overline{\gamma}})$, $q \in \mathrm{Crit}(f_{\underline{\gamma}})$. For $m \ge 0$ we denote by

$$\mathcal{M}_m^A(p, q; H, \{f_\gamma\}, J) \tag{35}$$

the union for $\gamma_1, \dots, \gamma_{m-1} \in \mathcal{P}_\lambda^{\le\alpha}$ and $A_1 + \ldots + A_m = A$ of the fibered products

$$W^u(p) \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_1}(S_{\overline{\gamma}}, S_{\gamma_1}) \times \mathbb{R}^+)_{\varphi_{f_{\gamma_1}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} (\mathcal{M}^{A_2}(S_{\gamma_1}, S_{\gamma_2}) \times \mathbb{R}^+)_{\varphi_{f_{\gamma_2}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}}$$
$$\ldots {}_{\varphi_{f_{\gamma_{m-1}}} \circ \underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \mathcal{M}^{A_m}(S_{\gamma_{m-1}}, S_{\underline{\gamma}})_{\underline{\mathrm{ev}}} \times W^s(q),$$

with the convention $\gamma_0 = \overline{\gamma}$. This is well defined as a smooth manifold of dimension

$$\begin{aligned}
&\dim \mathcal{M}_m^A(p,q;H,\{f_\gamma\},J) \\
&\quad = \quad \mathrm{ind}(p) - 1 + (\dim \mathcal{M}^{A_1}(S_{\overline{\gamma}}, S_{\gamma_1}) + 1) - 1 \\
&\qquad\qquad + (\dim \mathcal{M}^{A_2}(S_{\gamma_1}, S_{\gamma_2}) + 1) - 1 + \ldots \\
&\qquad\qquad + \dim \mathcal{M}^{A_m}(S_{\gamma_{m-1}}, S_{\underline{\gamma}}) - 1 + (1 - \mathrm{ind}(q)) \\
&\quad = \quad \mu(\overline{\gamma}) + \mathrm{ind}(p) - \mu(\underline{\gamma}) - \mathrm{ind}(q) + 2\langle c_1(TW), A_1 + \ldots + A_m\rangle - 1 \\
&\quad = \quad \mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) + 2\langle c_1(TW), A\rangle - 1.
\end{aligned}$$

The last equality follows from Lemma 3.4. Note that $\mathcal{M}_0^A(p,q;H,\{f_\gamma\},J)$ is naturally a submanifold of $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$. We denote

$$\mathcal{M}^A(p,q;H,\{f_\gamma\},J) := \bigcup_{m\geq 0} \mathcal{M}_m^A(p,q;H,\{f_\gamma\},J)$$

and we call this **the moduli space of Morse-Bott broken trajectories**, whereas $\mathcal{M}_m^A(p,q;H,\{f_\gamma\},J)$ is called **the moduli space of Morse-Bott broken trajectories with m sublevels** (see also Definition 4.1 and Figure 4).

Similarly, given $\overline{\gamma} \in \mathcal{P}_\lambda^{\leq\alpha}$, $p \in \mathrm{Crit}(f_{\overline{\gamma}})$, $\widetilde{q} \in \mathrm{Crit}(H)$, we define moduli spaces of Morse-Bott broken trajectories $\mathcal{M}_m^A(p,\widetilde{q};H,\{f_\gamma\},J)$, $m \geq 0$ and $\mathcal{M}^A(p,\widetilde{q};H,\{f_\gamma\},J)$ by replacing the last term $\mathcal{M}^{A_m}(S_{\gamma_{m-1}}, S_{\underline{\gamma}})_{\underline{\mathrm{ev}}} \times W^s(q)$ in the definition (35) with $\mathcal{M}^{A_m}(S_{\gamma_{m-1}}, \widetilde{q}; H, J)$. This is again well defined as a smooth manifold of dimension

$$\dim \mathcal{M}^A(p,\widetilde{q};H,\{f_\gamma\},J) = \mu(\overline{\gamma}_p) - \mu(\gamma_{\widetilde{q}}) + 2\langle c_1(TW), A\rangle - 1.$$

Again, $\mathcal{M}_0^A(p,\widetilde{q};H,\{f_\gamma\},J)$ is naturally a submanifold of $\mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$.

The significance of the above moduli spaces of broken Morse-Bott trajectories is explained by the following theorem, which describes the boundary of $\mathcal{M}^A_{]0,\delta_0[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)$ in (28) as $\delta \to 0$.

Theorem 3.7 (Correspondence Theorem). *Let $H \in \mathcal{H}'$ be fixed and let $\alpha := \lim_{t\to\infty} e^{-t}H(p,t)$ be the maximal slope of H. Let $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and $\{f_\gamma\} \in \mathcal{F}_{\mathrm{reg}}(H,J)$. There exists*

$$\delta_1 := \delta_1(H,J) \in]0, \delta_0[$$

such that, for any

$$\overline{\gamma}, \underline{\gamma} \in \mathcal{P}_\lambda^{\leq\alpha},\ p \in \mathrm{Crit}(f_{\overline{\gamma}}),\ q \in \mathrm{Crit}(f_{\underline{\gamma}}),$$

or

$$\overline{\gamma} \in \mathcal{P}_\lambda^{\leq\alpha},\ p \in \mathrm{Crit}(f_{\overline{\gamma}}),\ q \in \mathrm{Crit}(H),$$

and any $A \in H_2(W;\mathbb{Z})$ with

$$\mu(\overline{\gamma}_p) - \mu(\underline{\gamma}_q) + 2\langle c_1(TW), A\rangle = 1,$$

the following hold:

(i) J *is regular for* $\mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ *for all* $\delta \in]0, \delta_1[$*;*

(ii) *the space* $\mathcal{M}^A_{]0,\delta_1[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)$ *is a* 1*-dimensional manifold having a finite number of components which are graphs over* $]0, \delta_1[$*, i.e. the natural projection* $\mathcal{M}^A_{]0,\delta_1[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J) \to]0, \delta_1[$ *is a submersion;*

(iii) *there is a bijective correspondence between points*

$$[\mathbf{u}] \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$$

and connected components of $\mathcal{M}^A_{]0,\delta_1[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)$*.*

The proof of this statement, including a discussion of gluing and compactness for Morse-Bott moduli spaces, is given in Section 4.

We assume in the remainder of this section that the conclusions of Theorem 3.7 are satisfied. For each $[\mathbf{u}] \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ the sign $\epsilon(u_\delta)$ is constant on the corresponding connected component $C_{[\mathbf{u}]}$ for continuity reasons. We define a sign $\bar{\epsilon}(\mathbf{u})$ by

$$\bar{\epsilon}(\mathbf{u}) := \epsilon(u_\delta), \qquad \delta \in]0, \delta_1[, \quad (\delta, [u_\delta]) \in C_{[\mathbf{u}]}. \tag{36}$$

We define the **Morse-Bott chain groups** by

$$BC^a_*(H) \quad := \bigoplus_{\gamma \in \mathcal{P}_\lambda^{i^{-1}(a), \leq \alpha}} \Lambda_\omega \langle \gamma_{min}, \gamma_{Max} \rangle, \qquad a \neq 0, \tag{37}$$

$$BC^0_*(H) \quad := \bigoplus_{\widetilde{p} \in \mathrm{Crit}(H)} \Lambda_\omega \langle \widetilde{p} \rangle \ \oplus \bigoplus_{\gamma \in \mathcal{P}_\lambda^{i^{-1}(0), \leq \alpha}} \Lambda_\omega \langle \gamma_{min}, \gamma_{Max} \rangle. \tag{38}$$

where $\alpha := \lim_{t\to\infty} e^{-t} H(p,t)$ and $\mathcal{P}_\lambda^{i^{-1}(0), \leq \alpha} = \mathcal{P}_\lambda^{\leq \alpha} \cap \mathcal{P}_\lambda^{i^{-1}(0)}$. The grading is defined by

$$\begin{aligned} |e^A \widetilde{p}| &:= \mathrm{ind}(\widetilde{p}; -H) - n - 2\langle c_1(TW), A\rangle, \\ |e^A \gamma_{min}| &:= \mu(\gamma) + 1 - 2\langle c_1(TW), A\rangle, \\ |e^A \gamma_{Max}| &:= \mu(\gamma) - 2\langle c_1(TW), A\rangle. \end{aligned}$$

We define the **Morse-Bott differential**

$$\partial : BC^a_*(H) \to BC^a_{*-1}(H)$$

by

$$\partial \widetilde{p} \quad := \sum_{\substack{\widetilde{q} \in \mathrm{Crit}(H) \\ |\widetilde{p}| - |\widetilde{q}| = 1}} \ \sum_{[\mathbf{u}] \in \mathcal{M}^0(\widetilde{p}, \widetilde{q}; H, \{f_\gamma\}, J)} \bar{\epsilon}(\mathbf{u}) \widetilde{q}, \tag{39}$$

$$\begin{aligned} \partial \gamma_p \quad := & \sum_{\substack{\widetilde{q} \in \mathrm{Crit}(H) \\ |\gamma_p| - |e^A \widetilde{q}| = 1}} \ \sum_{[\mathbf{u}] \in \mathcal{M}^A(\gamma_p, \widetilde{q}; H, \{f_\gamma\}, J)} \bar{\epsilon}(\mathbf{u}) e^A \widetilde{q} \\ & + \sum_{\substack{\underline{\gamma} \in \mathcal{P}_\lambda^{\leq \alpha}, q \in \mathrm{Crit}(f_{\underline{\gamma}}) \\ |\gamma_p| - |e^A \underline{\gamma}_q| = 1}} \ \sum_{[\mathbf{u}] \in \mathcal{M}^A(\gamma_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)} \bar{\epsilon}(\mathbf{u}) e^A \underline{\gamma}_q, \qquad p \in \mathrm{Crit}(f_\gamma). \end{aligned} \tag{40}$$

The sums (39) and (40) clearly involve only periodic orbits in the same free homotopy class as that of $\widetilde{p}$ or γ_p respectively.

Remark 3.8. Since H is C^2-small, the moduli spaces $\mathcal{M}^A(\widetilde{p}, \widetilde{q}; H_\delta, J)$, $\widetilde{p}, \widetilde{q} \in \mathrm{Crit}(H)$ of expected dimension $\mathrm{ind}(\widetilde{p}; -H) - \mathrm{ind}(\widetilde{q}; -H) + 2\langle c_1(TW), A)\rangle - 1 = 0$ are independent of δ and consist exclusively of gradient trajectories of H in W [17, Theorem 6.1](see also [25, Theorem 7.3]). As a consequence, these moduli spaces are empty whenever $A \neq 0$.

We have, following directly from the definitions, an obvious isomorphism of free Λ_ω-modules

$$SC_*^a(H_\delta) \simeq BC_*^a(H), \qquad \delta \in]0, \delta_1[.$$

It follows now from Theorem 3.7 and the definition (36) of signs in the Morse-Bott complex that the corresponding differentials, defined by (14) and (39-40), also coincide. Here we use the fact that the Hamiltonian action functional decreases along Floer trajectories, hence the differential (14) applied to elements $\widetilde{p} \in \mathrm{Crit}(H)$ does not involve nonconstant elements of $\mathcal{P}(H_\delta)$ and reduces to (39) by Remark 3.8. As a consequence, we have

$$H_*(BC_*^a(H), \partial) = SH_*^a(H_\delta, J).$$

We shall construct in Section 4.4 a system of coherent orientations on the Morse-Bott moduli spaces

$$\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J), \qquad \mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)$$

whenever $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}_\lambda^{\leq \alpha}$ are good orbits. This in turn determines signs $\epsilon(\mathbf{u})$ via an orientation rule for fiber products (see (87)).

Proposition 3.9. *Assume* $\dim \mathcal{M}^A(p, q; H, \{f_\gamma\}, J) = 0$. *The bijective correspondence between elements* $[\mathbf{u}] \in \mathcal{M}_m^A(p, q; H, \{f_\gamma\}, J)$, $m \geq 1$ *and elements of* $[u_\delta] \in \mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ *given by Theorem 3.7 changes the signs by the rule*

$$\epsilon(\mathbf{u}) = (-1)^{m-1} \epsilon(u_\delta).$$

Moreover, if $m = 0$ *then* $\mathbf{u} = u_\delta$ *and* $\epsilon(\mathbf{u}) = \epsilon(u_\delta)$, p *is a minimum and* q *is a maximum, the moduli space* $\mathcal{M}_0^A(p, q; H, \{f_\gamma\}, J)$ *consists of the two gradient lines running from* p *to* q *and their signs are different if and only if the underlying orbit is good.*

In view of (36) and (39–40), this identification of signs between $\epsilon(\mathbf{u})$ and $\bar{\epsilon}(\mathbf{u})$ allows to define the Morse-Bott differential exclusively in terms of Morse-Bott data.

4 Morse-Bott moduli spaces

The structure of this section is as follows. We give in §4.1 the proof of Proposition 3.5, whereas Theorem 3.7 is proved in §4.2–§4.3, which treat compactness and gluing and correspond to assertions (i-ii) and (iii) respectively. Finally §4.4 contains a full discussion of orientation issues and the proof of Proposition 3.9.

4.1 Transversality

Proof of Proposition 3.5. We first prove (ii). Let $\mathcal{J}^\ell \subset \mathcal{J}$ be the space of admissible almost complex structures of class C^ℓ, $\ell \geq 1$, and let $\mathcal{J}^\ell(H) \subset \mathcal{J}^\ell$ be the set of almost complex structures J which, outside a fixed neighbourhood of the nonconstant periodic orbits of X_H, are independent of θ and satisfy $J\xi = \xi$, $J\frac{\partial}{\partial t} = R_\lambda$. By a standard trick of Taubes [15, Theorem 5.1] it is enough to show that there exists an open and dense set $\mathcal{J}^\ell_{\mathrm{reg}}(H) \subset \mathcal{J}^\ell(H)$ consisting of regular elements. We define the universal moduli spaces

$$\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, \mathcal{J}^\ell(H)) = \{(u, J) \mid J \in \mathcal{J}^\ell(H), u \in \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)\}$$

and

$$\mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, \mathcal{J}^\ell(H)) = \{(u, J) \mid J \in \mathcal{J}^\ell(H), u \in \mathcal{M}^A(S_{\overline{\gamma}}, \widetilde{q}; H, J)\}.$$

The main point is to show that these universal moduli spaces are Banach manifolds. Then the sets $\mathcal{J}^\ell_{\mathrm{reg}}(H)$ consist of the regular values of the natural projections from the universal moduli spaces to $\mathcal{J}^\ell(H)$. We only treat the case of $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, \mathcal{J}^\ell(H))$ since the second case is entirely similar, and we assume without loss of generality that $\overline{\gamma} \neq \underline{\gamma}$. This universal moduli space is the zero set of a distinguished section of a Banach vector bundle $\mathcal{E} \to \mathcal{B}^A \times \mathcal{J}^\ell(H)$ which we now define.

Let $p > 2$ and $d > 0$. Let $\mathcal{B}^A = \mathcal{B}^{1,p,d}(S_{\overline{\gamma}}, S_{\underline{\gamma}}, A; H)$ be the space of proper maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ which are locally in $W^{1,p}$ and satisfy

(i) the map u converges uniformly in θ as $s \to \pm\infty$ to $\underline{\gamma}(\cdot + \underline{\theta}_0)$, respectively $\overline{\gamma}(\cdot + \overline{\theta}_0)$, for some $\underline{\theta}_0, \overline{\theta}_0 \in S^1$, and represents the homology class $A \in H_2(W; \mathbb{Z})$;

(ii) there exist tubular neighbourhoods $\overline{U}$ and $\underline{U}$ of $\overline{\gamma}$ and $\underline{\gamma}$ respectively, together with parametrizations $\overline{\Psi} : \overline{U} \to S^1 \times \mathbb{R}^{2n-1}$ and $\underline{\Psi} : \underline{U} \to S^1 \times \mathbb{R}^{2n-1}$ such that

$$\begin{aligned} \overline{\Psi} \circ \overline{\gamma}(\theta) = \{\theta\} \times \{0\}, & \qquad \underline{\Psi} \circ \underline{\gamma}(\theta) = \{\theta\} \times \{0\}, \\ \overline{\Psi} \circ \overline{\gamma}(\theta + \overline{\theta}_0) - \overline{\Psi} \circ u(s, \theta) & \quad \in \quad W^{1,p}(]-\infty, -s_0], e^{d|s|} ds\, d\theta), \\ \underline{\Psi} \circ \underline{\gamma}(\theta + \underline{\theta}_0) - \underline{\Psi} \circ u(s, \theta) & \quad \in \quad W^{1,p}([s_0, \infty[, e^{d|s|} ds\, d\theta), \end{aligned}$$

for some $s_0 > 0$ sufficiently large.

Then $\mathcal{B}^A$ is a Banach manifold and, for d/p strictly smaller than the constant r in Proposition A.1, it contains the moduli spaces $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ for all $J \in \mathcal{J}^\ell$. Let $\mathcal{E} \to \mathcal{B}^A \times \mathcal{J}^\ell(H)$ be the Banach vector bundle with fiber $\mathcal{E}_{(u,J)} = L^p(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|} ds\, d\theta)$. Let $\bar{\partial}_H : \mathcal{B}^A \times \mathcal{J}^\ell(H) \to \mathcal{E}$ be the section defined by

$$\bar{\partial}_H(u, J) := \partial_s u + J_\theta(\partial_\theta u - X_H). \tag{41}$$

Then $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, \mathcal{J}^\ell(H)) = \bar{\partial}_H^{-1}(0)$ and it remains to show that $\bar{\partial}_H$ is transverse to the zero section. This means that the vertical differential

$$D\bar{\partial}_H(u, J) : T_u\mathcal{B}^A \times T_J\mathcal{J}^\ell(H) \to \mathcal{E}_{(u,J)}$$

is surjective for all $(u, J) \in \bar{\partial}_H^{-1}(0)$. We have

$$T_u\mathcal{B}^A = W^{1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta) \oplus \overline{V} \oplus \underline{V},$$

where $\overline{V}$, $\underline{V}$ are the one-dimensional real vector spaces generated by two sections of $u^*T\widehat{W}$ of the form $(1 - \beta(s, \theta))X_H(\overline{\gamma}(\theta))$ and $\beta(s, \theta)X_H(\underline{\gamma}(\theta))$ respectively, with $\beta(s, \theta) = \beta(s)$ a smooth cutoff function which vanishes for $s \le 0$ and is equal to 1 for $s \ge 1$. The space $T_J\mathcal{J}^\ell(H)$ consists of matrix valued functions $Y : S^1 \to \mathrm{End}(T\widehat{W})$ of class C^ℓ satisfying the conditions

$$J_\theta Y_\theta + Y_\theta J_\theta = 0, \qquad \widehat{\omega}(Y_\theta v, w) + \widehat{\omega}(v, Y_\theta w) = 0,\ \forall v, w \in T\widehat{W}, \tag{42}$$

and such that, outside fixed neighbourhoods of the nonconstant periodic orbits of X_H, they are independent of θ and have the form $\begin{pmatrix} Y_\xi & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the splitting $\xi \oplus \mathrm{Span}(R_\lambda, \frac{\partial}{\partial t})$. The operator $D\bar{\partial}_H(u, J)$ can be written

$$D\bar{\partial}_H(u, J) \cdot (\zeta, Y) = D_u\zeta + Y_\theta(u)(\partial_\theta u - X_H(u)).$$

Here

$$D_u : W^{1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta) \oplus \overline{V} \oplus \underline{V} \to L^p(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta)$$

is the linearization of the Cauchy-Riemann operator associated to the pair (H, J) and is explicitly given by formula (12). It is proved in [5, Proposition 4] that D_u is a Fredholm operator. It is at this point that the exponential weight plays a crucial role, due to the degeneracy of the asymptotic orbits. As a consequence the range of $D\bar{\partial}_H(u, J)$ is closed and we are left to prove that it is also dense. Let $q > 1$ be such that $1/p + 1/q = 1$. We show that every $\eta \in L^q(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta)$ satisfying

$$\int_{\mathbb{R}\times S^1} \langle \eta, D_u\zeta\rangle e^{d|s|}ds\, d\theta = 0, \qquad \int_{\mathbb{R}\times S^1} \langle \eta, Y_\theta(u)(\partial_\theta u - X_H(u))\rangle e^{d|s|}ds\, d\theta = 0 \tag{43}$$

for all ζ and Y vanishes. The first equation implies, by elliptic regularity, that η is of class C^ℓ and has the unique continuation property. Assume by contradiction that η does not vanish. Then the set $\{(s, \theta) : \eta(s, \theta) \neq 0\}$ is open and dense. On the other hand, it is proved in [15, Theorem 4.3] that the set

$$R(u) := \{(s, \theta) : \partial_s u(s, \theta) \neq 0,\ u(s, \theta) \neq \underline{\gamma}(\theta), \overline{\gamma}(\theta),\ u(s, \theta) \notin u(\mathbb{R} \setminus \{s\}, \theta)\}$$

of regular points of u is open and dense (although nondegeneracy of the asymptotic orbits is a standing assumption in [15], it does not play any role in the

proof of this result). Let $z_0=(s_0,\theta_0)$ be a point in $R(u)$ with $\eta(z_0)\neq 0$ and $u(z_0)$ belonging to the fixed open neighbourhood of $\overline{\gamma}$ (such a point exists since we have assumed $\overline{\gamma}\neq\underline{\gamma}$). One can choose a matrix $Y_{\theta_0}(u(z_0))$ satisfying (42) such that

$$\langle\eta(z_0),Y_{\theta_0}(u(z_0))J(u(z_0))\partial_s u(z_0)\rangle\neq 0.$$

Because z_0 is a regular point we can choose a time-dependent cutoff function $\rho:S^1\times\widehat{W}\to[0,1]$ supported near $(\theta_0,u(z_0))$ such that $Y:=\rho Y_{\theta_0}(u(z_0))$ satisfies

$$\int_{\mathbb{R}\times S^1}\langle\eta,Y_\theta(u)(\partial_\theta u-X_H(u))\rangle e^{d|s|}ds\,d\theta\neq 0.$$

This contradicts (43) and shows that $D\bar\partial_H(u,J)$ is surjective, hence the universal moduli space $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,\mathcal{J}^\ell(H))$ is a Banach manifold as claimed.

We now prove (i). The set $\mathcal{J}'(H)$ is obviously open. The fact that it is nonempty can be seen as follows. The space $S^1\times\mathbb{R}^2$ admits the "skating ring" contact form $\alpha=\sin\theta dx-\cos\theta dy$, $(\theta,x,y)\in S^1\times\mathbb{R}^2$ for which $\frac{\partial}{\partial\theta}\in\xi=\ker\alpha$. If J denotes the almost complex structure on ξ satisfying $J\frac{\partial}{\partial\theta}=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}$, then $[\frac{\partial}{\partial\theta},J\frac{\partial}{\partial\theta}]\neq 0$ and $[\frac{\partial}{\partial\theta},J\frac{\partial}{\partial\theta}]\notin\xi=\langle\frac{\partial}{\partial\theta},J\frac{\partial}{\partial\theta}\rangle$. This simple model can be adapted to our situation as follows. We can symplectically trivialize a neighbourhood of the simple orbit γ as $S^1\times\mathbb{R}^{2n-1}\ni(\theta,t,q_2,p_2,\ldots,q_n,p_n)$ with the standard symplectic form $d\theta\wedge dt+dq_2\wedge dp_2+\ldots+dq_n\wedge dp_n$, so that X_H corresponds to $\frac{\partial}{\partial\theta}$. Let J be a compatible almost complex structure such that $J\frac{\partial}{\partial\theta}=\frac{\partial}{\partial t}+\cos\theta\frac{\partial}{\partial q_2}+\sin\theta\frac{\partial}{\partial p_2}$. Since $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial t}$ commute we have $[X_H,JX_H]=[\frac{\partial}{\partial\theta},\cos\theta\frac{\partial}{\partial q_2}+\sin\theta\frac{\partial}{\partial p_2}]\neq 0$ and $[X_H,JX_H]\notin\langle X_H,JX_H\rangle$, so that $J\in\mathcal{J}'(H)$.

Let $\mathcal{J}'^\ell\subset\mathcal{J}'$ be the space of admissible almost complex structures of class C^ℓ, $\ell\geq 1$ which are independent of $\theta\in S^1$, and let $\mathcal{J}'^\ell(H)\subset\mathcal{J}'^\ell$ be the space of almost complex structures J which, outside a fixed neighbourhood of the nonconstant periodic orbits of X_H, satisfy $J\xi=\xi$, $J\frac{\partial}{\partial t}=R_\lambda$. It is enough to show that there exists an open and dense set $\mathcal{J}'^\ell_{\rm reg}(H)\subset\mathcal{J}'^\ell(H)$ consisting of elements which are regular for Floer trajectories with one nontrivial simple asymptote.

We have $\mathcal{J}'^\ell\subset\mathcal{J}^\ell$ and the main point is to show that the corresponding universal moduli spaces $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,\mathcal{J}'^\ell(H))\subset\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,\mathcal{J}^\ell(H))$ and $\mathcal{M}^A(S_{\overline{\gamma}},\widetilde{q};H,\mathcal{J}'^\ell(H))\subset\mathcal{M}^A(S_{\overline{\gamma}},\widetilde{q};H,\mathcal{J}^\ell(H))$ are Banach manifolds. We again treat only $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,\mathcal{J}'^\ell(H))$ and assume without loss of generality that $\overline{\gamma}$ is a simple orbit and $\overline{\gamma}\neq\underline{\gamma}$. This universal moduli space is the zero set of the section of the restricted bundle $\mathcal{E}\to\mathcal{B}^A\times\mathcal{J}'^\ell(H)$ defined by (41), and we have to show that the vertical differential $D\bar\partial_H(u,J):T_u\mathcal{B}^A\times T_J\mathcal{J}'^\ell(H)\to\mathcal{E}_{(u,J)}$ is surjective. Arguing by contradiction, we get an element $\eta\in L^q(\mathbb{R}\times S^1,u^*T\widehat{W};e^{d|s|}ds\,d\theta)$ of class C^ℓ which does not vanish on an open and dense subset of $\mathbb{R}\times S^1$ and satisfies $\int_{\mathbb{R}\times S^1}\langle\eta,Y(u)(\partial_\theta u-X_H(u))\rangle e^{d|s|}ds\,d\theta=0$ for any $Y\in T_J\mathcal{J}'^\ell(H)$. The main difference with respect to (ii) is that $T_J\mathcal{J}'^\ell(H)\subset T_J\mathcal{J}^\ell(H)$ consists of elements $Y\in\mathrm{End}(T\widehat{W})$ which are independent of θ.

Let

$$I(u) := \big\{(s,\theta)\,:\,\partial_s u(s,\theta)\neq 0,\ u^{-1}(u(s,\theta))=\{(s,\theta)\}\big\}$$

be the set of **injective** points, and denote $I_R(u) := I(u)\cap\,]R,\infty[\times S^1$, $R>0$. The main observation is that our special choice of $J\in\mathcal{J}'(H)$ implies that $I_R(u)$ is open and dense in $]R,\infty[\times S^1$ for R large enough. This is proved exactly as in [15, §7], and the main steps are the following. Since $\overline{\gamma}$ is simple, every u as above is **simple**, i.e. for every integer $m>1$ there exists a point $(s,\theta)\in\mathbb{R}\times S^1=\mathbb{R}\times\mathbb{R}/\mathbb{Z}$ such that $u(s,\theta+\frac{1}{m})\neq u(s,\theta)$. Let U be a neighbourhood of $\overline{\gamma}$ in which $[X_H,JX_H]\neq 0$ and $[X_H,JX_H]\notin\langle X_H,JX_H\rangle$. We call a point (s,θ) **regular** if $\partial_s u$, $\partial_\theta u$, $X_H(u)$, $JX_H(u)$ are linearly independent at (s,θ), and we denote by $R(u)$ the set of regular points. Then [15, Lemma 7.6] holds and [15, Lemma 7.7] shows that the set $\{(s,\theta)\in R(u)\,:\,u(s,\theta)\in U\}$ is open and dense in $u^{-1}(U)$. Note that we crucially use here our hypothesis $J\in\mathcal{J}'(H)$, which plays the role of the hypothesis $J\in\mathcal{J}_{\mathrm{ad}}(M,\omega,X)$ in [15]. Finally [15, Lemma 7.8] shows that the set of points which are regular and injective is open and dense in $u^{-1}(U)$, and in particular $I_R(u)$ is open and dense in $]R,\infty[\times S^1$ for R large enough.

We can then choose a point $z_0=(s_0,\theta_0)\in I_R(u)$ such that $\eta(z_0)\neq 0$ and a matrix $Y(u(z_0))$ satisfying (42) and $\langle\eta(z_0),Y(u(z_0))J(u(z_0))\partial_s u(z_0)\rangle\neq 0$. Since z_0 is an injective point we can choose a cutoff function $\rho:\widehat{W}\to\mathbb{R}$ supported near $u(z_0)$ such that $Y:=\rho Y(u(z_0))$ satisfies $\int_{\mathbb{R}\times S^1}\langle\eta,Y(u)(\partial_\theta u-X_H(u))\rangle e^{d|s|}ds\,d\theta\neq 0$. This contradiction shows that $D\bar\partial_H(u,J)$ is surjective and therefore $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,\mathcal{J}'^{\ell}(H))$ is a Banach manifold as claimed.

The dimension of the moduli space $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,J)$, $J\in\mathcal{J}_{\mathrm{reg}}(H)$ is equal to $\mathrm{ind}(D_u)-1$. The restriction of the operator D_u to the subspace $W^{1,p}(\mathbb{R}\times S^1,u^*T\widehat{W};e^{d|s|}ds\,d\theta)$ is conjugated to a Cauchy-Riemann operator

$$\mathcal{D}_u:W^{1,p}(\mathbb{R}\times S^1,u^*T\widehat{W};ds\,d\theta)\to L^p(\mathbb{R}\times S^1,u^*T\widehat{W};ds\,d\theta)$$

via multiplication by $e^{\frac{d}{p}|s|}$. If the asymptotics of D_u were nondegenerate, the Fredholm index of $\mathcal{D}_u$ would be given by [26]

$$\mu_{RS}(\overline{\gamma})-\mu_{RS}(\underline{\gamma})+2\langle c_1(TW),A\rangle.$$

Due to the one-dimensional degeneracy of $\overline{\gamma}$ and $\underline{\gamma}$, the actual index of $\mathcal{D}_u$ is obtained by a calculation analogous to [5, Proposition 4] (see also Lemma 3.4) :

$$(\mu_{RS}(\overline{\gamma})-\frac{1}{2})-(\mu_{RS}(\underline{\gamma})+\frac{1}{2})+2\langle c_1(TW),A\rangle. \tag{44}$$

We have proved in Lemma 3.4 that $\mu_{RS}(\gamma)=\mu(\gamma)+\frac{1}{2}$, hence

$$\mathrm{ind}(D_u)=\mathrm{ind}(\mathcal{D}_u)+2=\mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle+1.$$

Finally note that the evaluation maps $\overline{\mathrm{ev}},\underline{\mathrm{ev}}$ are well-defined and smooth on $\mathcal{B}^A$. Hence their restrictions to the moduli spaces are smooth as well. □

4.2 Compactness for Morse-Bott trajectories

Definition 4.1. *Let H, $\{f_\gamma\}$ and J be fixed as above, and let $p \in \mathrm{Crit}(f_{\overline{\gamma}})$, $q \in \mathrm{Crit}(f_{\underline{\gamma}})$. The space $\widehat{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)$ of* **parametrized Morse-Bott broken trajectories** *consists of tuples*

$$\mathbf{u} = (c_m, u_m, c_{m-1}, u_{m-1}, \dots, u_1, c_0)$$

such that

(i) $u_i \in \widehat{\mathcal{M}}^{A_i}(S_{\gamma_i}, S_{\gamma_{i-1}}; H, J)$, $i = 1, \dots, m$ *with* $\gamma_m = \overline{\gamma}$, $\gamma_0 = \underline{\gamma}$ *and* $A_1 + \dots + A_m = A$*;*

(ii) $c_0 : [-1, +\infty[\to S_{\gamma_0}$, $c_i : [-T_i/2, T_i/2] \to S_{\gamma_i}$, $i = 1, \dots, m-1$ *and* $c_m :]-\infty, 1] \to S_{\gamma_m}$ *satisfy* $\dot{c}_i = \nabla f_{\gamma_i} \circ c_i$, $i = 0, \dots, m$*;*

(iii) $\overline{\mathrm{ev}}(u_i) = \underline{\mathrm{ev}}(c_i)$, $\underline{\mathrm{ev}}(u_i) = \overline{\mathrm{ev}}(c_{i-1})$, $i = 1, \dots, m$ *and* $c_0(+\infty) = q$, $c_m(-\infty) = p$.

The space $\mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ of **unparametrized Morse-Bott broken trajectories** *consists of equivalence classes*

$$[\mathbf{u}] = (c_m, [u_m], c_{m-1}, [u_{m-1}], \dots, [u_1], c_0)$$

such that $\mathbf{u} \in \widehat{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)$.

Definition 4.2. *Let*

$$\mathbf{u}_k = (c_{m_k,k}, u_{m_k,k}, c_{m_k-1,k}, \dots, u_{1,k}, c_{0,k}) \in \widehat{\mathcal{M}}^A(p_k, q_k; H, \{f_\gamma\}, J)$$

with $k = 1, \dots, \ell$, and satisfying $q_k = p_{k-1}$ for $k = 2, \dots, \ell$. We denote $p := p_\ell$, $q := q_1$. A sequence $v_n \in \widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$ with $\delta_n \to 0$, $n \to \infty$ is said to **converge** *to $\overline{\mathbf{u}} := (\mathbf{u}_\ell, \dots, \mathbf{u}_1)$ if there exist shifts $(s^n_{i,k}) \in \mathbb{R}$, $i = 1, \dots, m_k$ such that*

$$v_n(\cdot + s^n_{i,k}, \cdot) \to u_{i,k}, \quad n \to \infty$$

uniformly on compact sets in $\mathbb{R} \times S^1$. We write in this case $v_n \to \overline{\mathbf{u}}$.

A sequence $[\widetilde{v}_n] \in \mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$ with $\delta_n \to 0$, $n \to \infty$ is said to **converge** *to $[\overline{\mathbf{u}}] \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ if there exist representatives v_n and $\overline{\mathbf{v}}$ such that $v_n \to \overline{\mathbf{v}}$ (this condition is obviously independent on the choice of representatives). We write in this case $[\widetilde{v}_n] \to [\overline{\mathbf{u}}]$.*

We call $\overline{\mathbf{u}}$ a **broken Floer trajectory with gradient fragments***. We call each of the $\mathbf{u}_k$'s a* **Floer trajectory with gradient fragments***. Each $\mathbf{u}_k$ is a* **level** *of $\overline{\mathbf{u}}$ and each $u_{i,k}$ is a* **sublevel** *of $\mathbf{u}_k$.*

Definition 4.3. *An element*

$$\mathbf{u} = (c_m, u_m, c_{m-1}, \dots, u_1, c_0) \in \widehat{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)$$

with $m \geq 1$ is **stable** *if each u_i, $i = 1, \dots, m$ is a nonconstant Floer trajectory and if each c_i, $i = 1, \dots, m-1$ defined on an interval of nonzero length is*

nonconstant. An element $\mathbf{u} = (c_0) \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ *is* **stable** *if* $p \neq q$. *A broken Floer trajectory with gradient fragments* $\overline{\mathbf{u}} = (\mathbf{u}_\ell, \ldots, \mathbf{u}_1)$ *is* **stable** *if each* $\mathbf{u}_k$, $k = 1, \ldots, \ell$ *is stable.*

Remark 4.4. A convergent sequence v_n of nonconstant Floer trajectories has a stable limit $\overline{\mathbf{u}}$ which is unique up to shifts on the $c_{i,k}$ and $u_{i,k}$.

The proofs of the next two lemmas use the asymptotic estimates proved in the Appendix. The relevant notation is introduced at the beginning of the Appendix, and we briefly recall it here for the reader's convenience. For each $\gamma \in \mathcal{P}(H)$ we choose coordinates $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ parametrizing a tubular neighbourhood of γ, such that $\vartheta \circ \gamma(\theta) = \theta$ and $z \circ \gamma(\theta) = 0$. Given a smooth function $f_\gamma : S_\gamma \to \mathbb{R}$, we denote by $\varphi_s^{f_\gamma}$ the gradient flow of f_γ with respect to the natural metric on S^1.

In a neighbourhood of $\gamma \in \mathcal{P}(H)$ the Floer equation $\partial_s u + J\partial_\theta u - JX_H = 0$ becomes $\partial_s Z + J\partial_\theta Z + J\frac{\partial}{\partial\vartheta} - JX_H = 0$, where $Z(s,\theta) := (\vartheta \circ u(s,\theta) - \theta, z \circ u(s,\theta))$. Since $X_H = \frac{\partial}{\partial\vartheta}$ on $\{z = 0\}$ this can be rewritten as $\partial_s Z + J\partial_\theta Z + Sz = 0$ for some matrix-valued function $S = S(\vartheta, z)$. The matrix $S_\infty(\theta) := S(\theta, 0)$ is symmetric. Let $A_\infty : H^k(S^1, \mathbb{R}^{2n}) \to H^{k-1}(S^1, \mathbb{R}^{2n})$ be the operator defined by $A_\infty Z := J\frac{d}{d\theta} Z + S_\infty(\theta) z$. The kernel of A_∞ has dimension one and is spanned by the constant vector $e_1 := (1, 0, \ldots, 0)$. We denote by Q_∞ the orthogonal projection onto $(\ker A_\infty)^\perp$ and we set $P_\infty := \mathbb{1} - Q_\infty$.

Lemma 4.5. *Let* $v_n \in \widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$ *with* $\delta_n \to 0$, $n \to \infty$ *and* $s_1^n < s_2^n$ *be shifts such that* $v_n(\cdot + s_1^n, \cdot) \to u_1$, $v_n(\cdot + s_2^n, \cdot) \to u_2$ *uniformly on compact sets, with* $u_1 \in \widehat{\mathcal{M}}^{A_1}(S_{\gamma_1}, S_\gamma; H, J)$ *and* $u_2 \in \widehat{\mathcal{M}}^{A_2}(S_\gamma, S_{\gamma_2}; H, J)$. *Any two sequences of shifts* $s_1^n < s_+^n < s_-^n < s_2^n$ *satisfying* $s_+^n - s_1^n \to \infty$, $s_2^n - s_-^n \to \infty$ *and*

$$\delta_n(s_+^n - s_1^n) \to 0, \qquad \delta_n(s_2^n - s_-^n) \to 0, \tag{45}$$

have the property that

$$v_n(\cdot + s_+^n, \cdot) \to \underline{\mathrm{ev}}(u_1), \qquad v_n(\cdot + s_-^n, \cdot) \to \overline{\mathrm{ev}}(u_2)$$

uniformly on compact sets.

Proof. We claim that there exists $K > 0$ such that $v_n([s_1^n + K, s_2^n - K] \times S^1)$ is contained in a given small neighbourhood of S_γ. If that was not the case, we could find a sequence $K_n \to \infty$ and a sequence $(s_n, \theta_n) \in [s_1^n + K_n, s_2^n - K_n] \times S^1$ such that $\mathrm{dist}(v_n(s_n, \theta_n), S_\gamma)$ is bounded away from zero. Up to a subsequence, $v_n(\cdot + s_n, \cdot)$ converges to some Floer trajectory v which must be nonconstant. On the other hand, for any $s \in \mathbb{R}$ and for any $K > 0$ we have, for n large enough,

$$\mathcal{A}_{H_{\delta_n}}\big(v_n(s + s_2^n - K, \cdot)\big) \le \mathcal{A}_{H_{\delta_n}}\big(v_n(s + s_n, \cdot)\big) \le \mathcal{A}_{H_{\delta_n}}\big(v_n(s + s_1^n + K, \cdot)\big),$$

and in the limit $\mathcal{A}_H(u_2(s - K, \cdot)) \le \mathcal{A}_H(v(s, \cdot)) \le \mathcal{A}_H(u_1(s + K, \cdot))$. We let K go to infinity and obtain $\mathcal{A}_H(\gamma) \le \mathcal{A}_H(v(s, \cdot)) \le \mathcal{A}_H(\gamma)$. This holds for all

$s \in \mathbb{R}$ and therefore the cylinder v is constant over some element of $\mathcal{P}(H)$, a contradiction which proves the claim.

By (98) in the proof of Proposition A.3 applied to v_n on $[s_1^n + K, s_2^n - K]$ we get

$$|Q_\infty v_n(s,\theta)| \le C \max(\|Q_\infty v_n(s_1^n + K)\|, \|Q_\infty v_n(s_2^n - K)\|). \tag{46}$$

Let γ_+ be the limit in S_γ of $v_n(s_+^n, \cdot)$, and let $I_n(\epsilon) := [s_+^n(\epsilon), s_+^n] \subset [s_1^n + K, s_+^n]$ be the maximal subinterval containing s_+^n such that $P_\infty v_n(s)$, $s \in I_n(\epsilon)$ is at distance at least ϵ from the critical points of f_γ, except maybe γ_+ (if the latter is a critical point). By the second part of Proposition A.3 applied to v_n on $I_n(\epsilon)$ we obtain

$$|\vartheta \circ v_n(s,\theta) - \theta - \varphi_{\delta s}^{f_\gamma}(\theta_0)| \le C \max(\|Q_\infty v_n(s_+^n(\epsilon))\|, \|Q_\infty v_n(s_+^n)\|) e^{M\delta_n(s_+^n - s_1^n)}.$$

Since $\delta_n(s_+^n - s_1^n) \to 0$ and taking into account (46) we get

$$|\vartheta \circ v_n(s,\theta) - \vartheta \circ \gamma_+(\theta)| \le C_1 \max(\|Q_\infty v_n(s_1^n + K)\|, \|Q_\infty v_n(s_2^n - K)\|). \tag{47}$$

For K large enough the right hand term becomes so small that the distance between $P_\infty v_n(s)$, $s \in I_n(\epsilon)$ and the critical points of f_γ, except possibly γ_+, is strictly bigger than ϵ, hence $I_n(\epsilon) = [s_1^n + K, s_+^n]$ by maximality (this holds for K large enough). Applying (47) to $s = s_1^n + K$ we obtain

$$|\vartheta \circ v_n(s_1^n + K, \theta) - \vartheta \circ \gamma_+(\theta)| \le C_1 \max(\|Q_\infty v_n(s_1^n + K)\|, \|Q_\infty v_n(s_2^n - K)\|).$$

Passing to the limit in the above inequality we obtain

$$|\vartheta \circ u_1(K,\theta) - \vartheta \circ \gamma_+(\theta)| \le C_1 \max(\|Q_\infty u_1(K)\|, \|Q_\infty u_2(-K)\|).$$

Letting $K \to \infty$ we obtain $\underline{\mathrm{ev}}(u_1) = \gamma_+$. That this implies uniform convergence on compact sets to the constant cylinder over $\underline{\mathrm{ev}}(u_1)$ can be seen in two ways: either one notices that the above estimates hold uniformly when s_+^n is replaced with $s_+^n + K_+$, where K_+ is a bounded constant, or one uses the fact that a Floer trajectory passing through a periodic orbit is necessarily a constant cylinder, by unique continuation applied to the infinite jet at that orbit [21, Theorem 2.3.2].

A similar argument proves the assertion involving $\overline{\mathrm{ev}}(u_2)$. $\square$

Lemma 4.6. *Let $v_n \in \widehat{\mathcal{M}}^A(p,q;H_{\delta_n},J)$ with $\delta_n \to 0$, $n \to \infty$. Assume we are given two sequences of shifts $s_1^n < s_2^n$ such that $v_n(\cdot + s_i^n, \cdot)$, $i = 1,2$ converge uniformly on compact sets to constant cylinders u_{γ_i} over orbits γ_i belonging to the same family S_γ. Then there exists a (possibly broken) gradient trajectory of f_γ starting at γ_1 and ending at γ_2. Moreover, the length of the gradient trajectory is $T = \lim_{n\to\infty} \delta_n(s_2^n - s_1^n)$.*

Proof. We claim that, for n large enough, $v_n([s_1^n, s_2^n] \times S^1)$ is entirely contained in an arbitrarily small neighbourhood of γ. By contradiction, if this fails we can reparametrize the sequence v_n so that it converges to a nonconstant Floer

trajectory $v \in \mathcal{M}^B(\overline{\gamma}', \underline{\gamma}'; H, J)$ for some class $B \in H_2(M;\mathbb{Z})$ and some $\overline{\gamma}', \underline{\gamma}' \in \mathcal{P}(H)$ such that their actions satisfy $\mathcal{A}_H(\gamma) \le \mathcal{A}_H(\underline{\gamma}') < \mathcal{A}_H(\overline{\gamma}') \le \mathcal{A}_H(\gamma)$, which is impossible.

We first assume that γ_1 is not a critical point of f_γ. Let $\epsilon > 0$ be fixed and denote by $I_n(\epsilon) = [s_1^n, s_2^n(\epsilon)] \subset [s_1^n, s_2^n]$ the maximal subinterval containing s_1^n such that the distance between $P_\infty v_n(s)$, $s \in I_n(\epsilon)$ and $\mathrm{Crit}(f_\gamma)$ is at least ϵ. We can apply Proposition A.3 to v_n and $I_n(\epsilon)$. In particular, for some sequence $\theta_n \in S^1$ we have

$$\lim_{n\to\infty} \sup_{(s,\theta)\in I_n(\epsilon)\times S^1} |\vartheta \circ v_n(s,\theta) - \theta - \varphi_s^{\delta_n f_\gamma}(\theta_n)| = 0.$$

Since $v_n(s_1^n, \cdot)$ converges to γ_1, we also have

$$\lim_{n\to\infty} \sup_{(s,\theta)\in I_n(\epsilon)\times S^1} |\vartheta \circ v_n(s,\theta) - \theta - \varphi^{f_\gamma}_{\delta_n(s-s_1^n)}(\gamma_1)| = 0. \tag{48}$$

Modulo passing to a subsequence we know that $v_n(s_2^n(\epsilon), \cdot)$ converges, which together with (48) implies that $\delta_n(s_2^n(\epsilon) - s_1^n)$ converges to $T(\epsilon) \in \mathbb{R}_+$. This holds for each $\epsilon > 0$ and, since $s_2^n(\epsilon) < s_2^n(\epsilon')$ if $\epsilon > \epsilon'$, the limit $\lim_{\epsilon\to 0} T(\epsilon) = T \in \overline{\mathbb{R}}_+$ exists. Then $\varphi_s^{f_\gamma}(\gamma_1)$, $s \in [0, T]$ is a gradient trajectory starting at γ_1.

If T is finite then this trajectory, and therefore $v_n(I_n(\epsilon) \times S^1)$ stay at a fixed distance from $\mathrm{Crit}(f_\gamma)$ for n large enough. Hence $I_n(\epsilon) = I_n$ for ϵ sufficiently small and we are done. If T is infinite and the limit $\lim_{s\to\infty} \varphi_s^{f_\gamma}(\gamma_1)$ is equal to γ_2, we are also done. Otherwise we are in the next case, with shifts $\widetilde{s}_1^n := \lim_{\epsilon\to 0} s_2^n(\epsilon)$ and $\widetilde{s}_2^n := s_2^n$.

We now assume that γ_1 is a critical point of f_γ and $\gamma_1 \ne \gamma_2$. Given $\epsilon > 0$ we denote by $I_n(\epsilon) = [s_1^n, s_2^n(\epsilon)] \subset [s_1^n, s_2^n]$ the maximal subinterval containing s_1^n such that the distance between $P_\infty v_n(s)$, $s \in I_n(\epsilon)$ and $\mathrm{Crit}(f_\gamma) \setminus \{\gamma_1\}$ is at least ϵ. For $\epsilon > 0$ small enough the loops $P_\infty v_n(s_2^n(\epsilon))$ are at a distance bigger than ϵ from γ_1 and, up to a subsequence, $v_n(s_2^n(\epsilon), \cdot)$ converges to some $\widetilde{\gamma}_2 \in S_\gamma$ which is not a critical point of f_γ. The same argument as in the previous case applied "backwards" to the shifts $s_1^n < s_2^n(\epsilon)$ produces a negative gradient trajectory running from $\widetilde{\gamma}_2$ to some critical point $\widetilde{\gamma}_1$. By definition of $I_n(\epsilon)$, we must have $\widetilde{\gamma}_1 = \gamma_1$ and we thus obtain a gradient trajectory from γ_1 to $\widetilde{\gamma}_2$. We are now in the first case with shifts $\widetilde{s}_1^n := s_2^n(\epsilon)$ and $\widetilde{s}_2^n := s_2^n$.

We successively apply the above two cases in order to produce a broken gradient trajectory from γ_1 to γ_2. This is a finite process since a broken trajectory has a finite number of nonconstant fragments. $\square$

Proposition 4.7. *Let $v_n \in \widehat{\mathcal{M}}^A(p, q; H_{\delta_n}, J)$ with $\delta_n \to 0$, $n \to \infty$. There exists a broken Floer trajectory with gradient fragments $\overline{\mathbf{u}}$ and a subsequence (still denoted by v_n) such that $v_n \to \overline{\mathbf{u}}$.*

Proof. The energy $\mathcal{E}(v_n) := \mathcal{E}_{J,H_{\delta_n}}(v_n)$ defined in (11) satisfies

$$\mathcal{E}(v_n) = -\int_{S^1} H_{\delta_n}(\theta, \overline{\gamma}_p(\theta))\, d\theta + \int_{S^1} H_{\delta_n}(\theta, \underline{\gamma}_q(\theta))\, d\theta.$$

Since $H_{\delta_n} \to H$ we infer that $\mathcal{E}(v_n)$ is uniformly bounded.

Floer's compactness theorem [12, Proposition 3c] applies to our situation and provides a collection of Floer trajectories u_i, $i = 1, \ldots, m$ for the pair (H, J) together with holomorphic spheres attached to them, as well as shifts (s_i^n) such that $v_n(\cdot + s_i^n, \cdot)$ converges to u_i and its associated holomorphic spheres in the sense of nodal curves. Condition (1) implies symplectic asphericity $\langle \omega, \pi_2(\widehat{W}) \rangle = 0$, therefore holomorphic spheres in $(\widehat{W}, J)$ are constant and the shifted v_n converge to u_i uniformly on compact sets.

Because the action spectrum of ∂W was assumed to be discrete and injective the trajectories u_i connect with each other, in the sense that $\overline{\mathrm{ev}}(u_i)$ and $\underline{\mathrm{ev}}(u_{i+1})$ belong to the same family of trajectories S_{γ_i}, $i = 1, \ldots, m-1$. Moreover, $\overline{\mathrm{ev}}(u_m)$ belongs to $S_{\overline{\gamma}}$ and $\underline{\mathrm{ev}}(u_1)$ belongs to $S_{\underline{\gamma}}$.

By Lemma 4.5 there exist shifts $s_{i,\pm}^n$ such that $v_n(\cdot + s_{i,+}^n, \cdot)$ converges to the constant cylinder over $\underline{\mathrm{ev}}(u_i)$, and $v_n(\cdot + s_{i,-}^n, \cdot)$ converges to the constant cylinder over $\overline{\mathrm{ev}}(u_i)$. Applying Lemma 4.6 with shifts $s_{i,+}^n < s_{i-1,-}^n$, $i = 2, \ldots, m$ and n large enough, we obtain broken gradient trajectories c_{i-1} starting at $\underline{\mathrm{ev}}(u_i)$ and ending at $\overline{\mathrm{ev}}(u_{i-1})$. Let now s_-^n, s_+^n be shifts such that $v_n(\cdot + s_-^n, \cdot) \to p$ and $v_n(\cdot + s_+^n, \cdot) \to q$. Applying Lemma 4.6 with shifts $s_-^n < s_{m,-}^n$ and with shifts $s_{1,+}^n < s_+^n$ we obtain broken gradient trajectories c_m starting at p and ending at $\overline{\mathrm{ev}}(u_m)$ and c_0 starting at $\underline{\mathrm{ev}}(u_1)$ and ending at q. Since all S_γ are circles of periodic orbits, the broken gradient trajectories c_i, $i = 0, \ldots, m$ consist each of a single fragment.

The construction of a stable broken Floer trajectory with gradient fragments out of the data c_i, u_i is straightforward and goes as follows. The collection of points of the form $\underline{\mathrm{ev}}(u_{i+1})$, $\overline{\mathrm{ev}}(u_i)$ which are critical points of f_{γ_i} determine a partition

$$(c_{m_\ell,\ell}, u_{m_\ell,\ell}, c_{m_\ell-1,\ell}, \ldots, c_{1,\ell}, u_{1,\ell}, c_{0,\ell}), \ldots, (c_{m_1,1}, u_{m_1,1}, \ldots, u_{1,1}, c_{0,1})$$

of the ordered tuple $(c_m, u_m, \ldots, c_1, u_1, c_0)$. Note that the $c_{m_k,k}$ and $c_{0,k}$ may either be missing or be constant and exactly one of $c_{0,k}$ and $c_{m_{k-1},k-1}$ is missing. In such a situation we set $c_{m_k,k}$ or $c_{0,k}$ to be a constant trajectory at the relevant critical point, defined on a semi-infinite interval. □

4.3 Gluing for Morse-Bott moduli spaces

We prove in this subsection the assertions (i-ii) of Theorem 3.7. The following notation was introduced in the previous subsection. For $\gamma \in \mathcal{P}(H)$ we choose coordinates $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ parametrizing a tubular neighbourhood of γ, such that $\vartheta \circ \gamma(\theta) = \theta$ and $z \circ \gamma(\theta) = 0$. Given a smooth function $f_\gamma : S_\gamma \to \mathbb{R}$, we denote by $\varphi_s^{f_\gamma}$ the gradient flow of f_γ with respect to the natural metric on S^1. The orthogonal projection onto the 1-dimensional kernel of the asymptotic operator at $\gamma \in \mathcal{P}(H)$ is denoted by P_∞, and we denote $Q_\infty := \mathbb{1} - P_\infty$.

Let $p > 2, d > 0$ and $\delta > 0$. Let $\mathcal{B}_\delta^A = \mathcal{B}_\delta^{1,p,d}(\overline{\gamma}_p, \underline{\gamma}_q, A; H, \{f_\gamma\})$ be the space of proper maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ which are locally in $W^{1,p}$ and satisfy

(i) the map u converges uniformly in θ as $s \to \pm\infty$ to $\underline{\gamma}_q$, respectively $\overline{\gamma}_p$, and represents the homology class $A \in H_2(W;\mathbb{Z})$;

(ii) there exist tubular neighbourhoods $\overline{U}$ and $\underline{U}$ of $\overline{\gamma}$ and $\underline{\gamma}$ respectively, parametrized by $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ such that

$$\begin{aligned}
\vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_{\overline{\gamma}}}(\overline{\theta}_0) &\in W^{1,p}(]-\infty,-s_0] \times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta),\\
z \circ u(s,\theta) &\in W^{1,p}(]-\infty,-s_0] \times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta),\\
\vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_{\underline{\gamma}}}(\underline{\theta}_0) &\in W^{1,p}([s_0,\infty[\times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta),\\
z \circ u(s,\theta) &\in W^{1,p}([s_0,\infty[\times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta),
\end{aligned}$$

for some $s_0 > 0$ sufficiently large and some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ satisfying

$$\lim_{s\to-\infty} \varphi_s^{f_\gamma}(\overline{\theta}_0) = p, \qquad \lim_{s\to+\infty} \varphi_s^{f_\gamma}(\underline{\theta}_0) = q. \tag{49}$$

Then $\mathcal{B}_\delta^A$ is a Banach manifold and, for $d > 0$ sufficiently small, it contains the moduli spaces $\mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ for all $J \in \mathcal{J}$ (see Proposition A.2 in the Appendix). Let $\mathcal{E} \to \mathcal{B}_\delta^A$ be the Banach vector bundle with fiber $\mathcal{E}_{(u,J)} = L^p(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|} ds\, d\theta)$. Let $\bar\partial_{H_\delta,J} : \mathcal{B}_\delta^A \to \mathcal{E}$ be the section defined by

$$\bar\partial_{H_\delta,J}(u) := \partial_s u + J_\theta(\partial_\theta u - X_{H_\delta}).$$

Then $\mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J) = \bar\partial^{-1}_{H_\delta,J}(0)$. From now on we fix $J \in \mathcal{J}_{\rm reg}(H)$. In order to prove (i) in Theorem 3.7 we need to show that the vertical differential $D_u : T_u\mathcal{B}_\delta^A \to \mathcal{E}_u$ defined by (12) is surjective for all $u \in \mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ when $\delta > 0$ is sufficiently small and the expected dimension of the moduli space is zero. We have

$$T_u\mathcal{B}_\delta^A = W^{1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|} ds\, d\theta) \oplus \overline{V}_u \oplus \underline{V}_u,$$

where $\overline{V}_u$, $\underline{V}_u$ are real vector spaces of dimension

$$\dim \overline{V}_u = \operatorname{ind}(p), \qquad \dim \underline{V}_u = 1 - \operatorname{ind}(q).$$

When their dimension is nonzero $\overline{V}_u$ and $\underline{V}_u$ are respectively generated by two sections of $u^*T\widehat{W}$ of the form

$$(1-\beta(s,\theta))\nabla f_{\overline{\gamma}}(\vartheta \circ u(s,0)) \quad \text{and} \quad \beta(s,\theta)\nabla f_{\underline{\gamma}}(\vartheta \circ u(s,0)),$$

with $\beta(s,\theta) = \beta(s)$ a smooth cutoff function which vanishes for $s \le 0$ and is equal to 1 for $s \ge 1$. The fact that $\overline{V}_u$ and $\underline{V}_u$ have varying dimensions is a consequence of condition (49).

We shall prove surjectivity of D_u by showing that the elements of the moduli space $\mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ can be approximated, for $\delta > 0$ small enough, by gluing the elements of $\mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ with fragments of gradient trajectories of the Morse functions f_γ.

Given $a, b \in \overline{\mathbb{R}}$, $a < b$ we define intervals

$$I(a,b) = \begin{cases} [a,b] & \text{if } a,b \in \mathbb{R}, \\ ]-\infty, b] & \text{if } a = -\infty,\ b \in \mathbb{R}, \\ [a, \infty[& \text{if } a \in \mathbb{R},\ b = \infty. \end{cases}$$

For $b - a > 4$ and $|\epsilon| < 1$, we let $h_{a,b,\epsilon} : \mathbb{R} \to I(a,b) \subset \mathbb{R}$ be a collection of smooth increasing functions such that $h_{a,b,\epsilon}(s) = a$ if $s \le a - \epsilon/2$, $h_{a,b,\epsilon}(s) = b$ if $s \ge b + \epsilon/2$ and $h_{a,b,\epsilon}(s) := s$ if $a - \epsilon/2 + 1 < s < b + \epsilon/2 - 1$. We can of course make the family $\{h_{a,b,\epsilon}\}$ depend smoothly on a, b and ϵ. We define $k_{a,b,\epsilon}(s) := \frac{d}{d\sigma}|_{\sigma=0} h'_{a-\sigma,b+\sigma,\epsilon}(s)$. The support of $k_{a,b,\epsilon}$ is contained in $[a - \epsilon/2, a - \epsilon/2 + 1] \cup [b + \epsilon/2 - 1, b + \epsilon/2]$. We may assume without loss of generality that $h'_{a,b,\epsilon}$ and $k_{a,b,\epsilon}$ are uniformly bounded.

Convention. If $\epsilon = 0$ we shall omit it from all subsequent decorations, and we set $\epsilon = 0$ if $a = -\infty$ or $b = +\infty$.

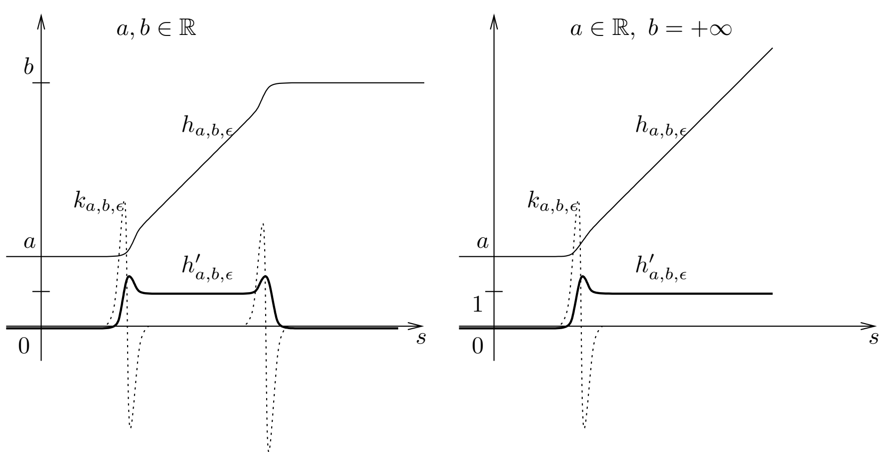

Figure 2: The reparametrization function $h_{a,b,\epsilon}$ and its derivatives.

Let $\gamma \in \mathcal{P}_\lambda$ and $c : I(a,b) \to S_\gamma \subset \widehat{W}$ be a fragment of gradient trajectory for the function f_γ, i.e. $\dot{c} = \nabla f_\gamma \circ c$. We define the corresponding **gradient cylinder**

$$u_{\delta,\gamma,a,b,\epsilon} : \mathbb{R} \times S^1 \to S_\gamma \subset \widehat{W}$$

by the equation

$$\vartheta \circ u_{\delta,\gamma,a,b,\epsilon}(s,\theta) = \vartheta \circ c(\delta h_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}(s)) + \theta. \tag{50}$$

Then $\lim_{s\to-\infty} \vartheta \circ u_{\delta,\gamma,a,b,\epsilon}(s,\theta) = \vartheta \circ c(a) + \theta$ and $\lim_{s\to+\infty} \vartheta \circ u_{\delta,\gamma,a,b,\epsilon}(s,\theta) = \vartheta \circ c(b) + \theta$.

For $\gamma \in \mathcal{P}_\lambda$ we define Banach manifolds $\mathcal{B}_\delta^{1,p,d}(S_\gamma, S_\gamma; f_\gamma)$, $\mathcal{B}_\delta^{1,p,d}(p, S_\gamma; f_\gamma)$, $p \in \mathrm{Crit}(f_\gamma)$ and $\mathcal{B}_\delta^{1,p,d}(S_\gamma, q; f_\gamma)$, $q \in \mathrm{Crit}(f_\gamma)$ consisting of maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ which are locally of class $W^{1,p}$, whose asymptotics are translates of γ, which represent the zero homology class and which satisfy the following asymptotic conditions.

(i) for $\mathcal{B}_\delta^{1,p,d}(S_\gamma, S_\gamma; f_\gamma)$: there exists a neighbourhood U of S_γ together with a parametrization $(\vartheta, z) : U \to S^1 \times \mathbb{R}^{2n-1}$ such that

$$\begin{array}{rcl} \vartheta \circ u(s,\theta) - \theta - \overline{\theta}_0 & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \\ \vartheta \circ u(s,\theta) - \theta - \underline{\theta}_0 & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \end{array}$$

for some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ and some $s_0 > 0$. Moreover, there exists $T > 0$ such that

$$\varphi_T^{f_\gamma}(\overline{\theta}_0) = \underline{\theta}_0; \tag{51}$$

(ii) for $\mathcal{B}_\delta^{1,p,d}(p, S_\gamma; f_\gamma)$: there exists a neighbourhood U of S_γ parametrized by $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ such that

$$\begin{array}{rcl} \vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_\gamma}(\overline{\theta}_0) & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \\ \vartheta \circ u(s,\theta) - \theta - \underline{\theta}_0 & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \end{array}$$

for some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ such that $\lim_{s\to-\infty} \varphi_s^{f_\gamma}(\overline{\theta}_0) = \lim_{s\to-\infty} \varphi_s^{f_\gamma}(\underline{\theta}_0) = p$ and some $s_0 > 0$;

(iii) for $\mathcal{B}_\delta^{1,p,d}(S_\gamma, q; f_\gamma)$: there exists a neighbourhood U of S_γ parametrized by $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ such that

$$\begin{array}{rcl} \vartheta \circ u(s,\theta) - \theta - \overline{\theta}_0 & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \\ \vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_\gamma}(\underline{\theta}_0) & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}; e^{d|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}^{2n-1}; e^{d|s|} ds\, d\theta), \end{array}$$

for some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ such that $\lim_{s\to\infty} \varphi_s^{f_\gamma}(\underline{\theta}_0) = \lim_{s\to\infty} \varphi_s^{f_\gamma}(\overline{\theta}_0) = q$ and some $s_0 > 0$.

We will designate one of the above three spaces by $\mathcal{B}'_\delta$. We define evaluation maps $\overline{\mathrm{ev}}$ and $\underline{\mathrm{ev}}$ on $\mathcal{B}'_\delta$ by

$$\overline{\mathrm{ev}}(u) = \lim_{s\to-\infty} u(s,\cdot), \qquad \underline{\mathrm{ev}}(u) = \lim_{s\to+\infty} u(s,\cdot).$$

Any map $u = u_{\delta,\gamma,a,b,\epsilon}$ belongs to a suitable space $\mathcal{B}'_\delta$, depending on a, b being finite or not. The tangent space $T_u\mathcal{B}'_\delta$ has a natural decomposition

$$T_u\mathcal{B}'_\delta = W^{1,p,d}(\mathbb{R}\times S^1, u^*T\widehat{W}) \oplus \overline{V}'_u \oplus \underline{V}'_u, \tag{52}$$

where $\overline{V}'_u, \underline{V}'_u$ are real vector spaces of dimensions

$$\dim \overline{V}'_u = \begin{cases} 1 & \text{if } a\in\mathbb{R}, \\ \mathrm{ind}(p) & \text{if } a=-\infty, \end{cases} \quad \dim \underline{V}'_u = \begin{cases} 1 & \text{if } b\in\mathbb{R}, \\ 1-\mathrm{ind}(q) & \text{if } b=+\infty. \end{cases} \tag{53}$$

When the dimensions are respectively nonzero the generators of $\overline{V}'_u, \underline{V}'_u$ are sections given as follows.

(i) for $\mathcal{B}^{1,p,d}_\delta(S_\gamma, S_\gamma; f_\gamma)$ the sections are

$$(1-\beta(s,\theta))X_H(\gamma(\theta+\overline{\theta}_0)) \text{ and } \beta(s,\theta)X_H(\gamma(\theta+\underline{\theta}_0));$$

(ii) for $\mathcal{B}^{1,p,d}_\delta(p, S_\gamma; f_\gamma)$ the sections are

$$(1-\beta(s,\theta))\nabla f_\gamma(\vartheta\circ u(s,0)) \text{ and } \beta(s,\theta)X_H(\gamma(\theta+\underline{\theta}_0));$$

(iii) for $\mathcal{B}^{1,p,d}_\delta(S_\gamma, q; f_\gamma)$ the sections are

$$(1-\beta(s,\theta))X_H(\gamma(\theta+\overline{\theta}_0)) \text{ and } \beta(s,\theta)\nabla f_\gamma(\vartheta\circ u(s,0)).$$

We recall that $\beta(s,\theta)=\beta(s)$ is a smooth cutoff function which vanishes for $s\le 0$ and is equal to 1 for $s\ge 1$. The norm on $T_u\mathcal{B}'_\delta$ is chosen such that the norm of the above generators of $\overline{V}'_u$, $\underline{V}'_u$ is equal to 1. Let $\mathcal{E}\to\mathcal{B}'_\delta$ be the Banach vector bundle with fiber

$$\mathcal{E}_u = L^p(\mathbb{R}\times S^1, u^*T\widehat{W}; e^{d|s|}ds\,d\theta).$$

We are interested in the family of sections $\bar\partial_{a,b,\epsilon} := \bar\partial_{H'_{a,b,\epsilon},J} : \mathcal{B}'_\delta \to \mathcal{E}$, with

$$\begin{aligned} H'_{a,b,\epsilon} &= H + h'_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}(s)(H_\delta - H) \\ &= H + \delta h'_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}(s)\rho f_\gamma(\ell_\gamma\vartheta - \ell_\gamma\theta). \end{aligned} \tag{54}$$

Here we use the definition (25) of H_δ. This is a three-parameter family in case (i) and a two-parameter family in cases (ii) and (iii). Its main feature is that

$$\bar\partial_{a,b,\epsilon}(u_{\delta,\gamma,a,b,\epsilon}) = 0.$$

We note that neither of the operators $\bar\partial_{H,J}$ and $\bar\partial_{H_\delta,J}$ defines a section $\mathcal{B}'_\delta\to\mathcal{E}$ if a or b is infinite. The vertical differential $D_u := D_u^{a,b,\epsilon} : T_u\mathcal{B}'_\delta \to \mathcal{E}_u$ of each of the sections $\bar\partial_{a,b,\epsilon}$ is given by formula (12) and is a Fredholm operator whose index has the following values (see also (44)).

(i) for $\mathcal{B}_\delta^{1,p,d}(S_\gamma, S_\gamma; f_\gamma)$

$$\mathrm{ind}(D_u) = (\mu_{RS}(\gamma) - \frac{1}{2}) - (\mu_{RS}(\gamma) + \frac{1}{2}) + 2 = 1,$$

(ii) for $\mathcal{B}_\delta^{1,p,d}(p, S_\gamma; f_\gamma)$

$$\mathrm{ind}(D_u) = (\mu_{RS}(\gamma) - \frac{1}{2}) - (\mu_{RS}(\gamma) + \frac{1}{2}) + \mathrm{ind}(p) + 1 = \mathrm{ind}(p),$$

(iii) for $\mathcal{B}_\delta^{1,p,d}(S_\gamma, q; f_\gamma)$

$$\mathrm{ind}(D_u) = (\mu_{RS}(\gamma) - \frac{1}{2}) - (\mu_{RS}(\gamma) + \frac{1}{2}) + 1 + 1 - \mathrm{ind}(q) = 1 - \mathrm{ind}(q).$$

In formulas (ii) and (iii) the asymptotics of the operator obtained by conjugation with $e^{\frac{d}{p}|s|}$ do not depend on $\mathrm{ind}(p)$, $\mathrm{ind}(q)$ because, for δ small, the exponential weight $\frac{d}{p}$ overrides the contribution of the perturbation $H_\delta - H$.

Proposition 4.8. *Let $u = u_{\delta,\gamma,a,b,\epsilon} \in \mathcal{B}'_\delta$. The operator*

$$D_u : W^{1,p}(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}) \oplus \overline{V}'_u \oplus \underline{V}'_u \to L^p(\mathbb{R} \times S^1, u^*T\widehat{W}; e^{d|s|}dsd\theta)$$

is surjective for $\delta > 0$ small enough.

Proof. In order to compute D_u we choose ∇ to be the Levi-Civita connection corresponding to a (split) metric given by $(d\lambda + dt \wedge \lambda)(\cdot, J\cdot)$. It is a general fact that the operator D_u can be written in a unitary trivialization of $u^*T\widehat{W}$ as

$$(D_u\zeta)(s,\theta) = \partial_s\zeta + J_0\partial_\theta\zeta + S(s,\theta)\zeta(s,\theta),$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}$ and S is asymptotically symmetric as $s \to \pm\infty$. We can choose the trivialization so that X_H and $\partial/\partial t$ correspond to constant vectors in $\mathbb{R}^{2n}$. We denote $\overline{S} := \lim_{s\to-\infty} S(s,\cdot)$. In this situation the matrix S has the following properties:

(i) $\|S(s,\theta) - \overline{S}(\vartheta \circ u(s,\theta) - \vartheta \circ c(a))\|$ is bounded by a constant multiple of δ. This is because, for $s \in \mathbb{R}$, the restriction of u to $[s-1, s+1] \times S^1$ is δ-close to the constant cylinder over the orbit $u(s,\cdot) \in S_\gamma$.

(ii) the action of $S(s,\theta)$ on the (constant) vector of $\mathbb{R}^{2n}$ corresponding to X_H is multiplication by

$$\delta k(s) := \delta h'_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}(s) f''_\gamma(\vartheta(u(s,\theta)) - \theta),$$

and this expression goes to zero with δ.

(iii) the matrix $S(s,\theta)$ sends the subspace corresponding to ξ to itself and sends $\partial/\partial t$ on a multiple of the form $(E + \delta F(s,\theta))\partial/\partial t$, with $E > 0$ and F a bounded function of (s,θ). This follows from (12), (24) and the fact that $\nabla_{\partial/\partial t} R_\lambda = 0$ and $\nabla_v R_\lambda \in \xi$, $v \in \xi$;

(iv) there is a constant $C > 0$ such that $\|S'(s,\theta)\| \le C\delta$ for all $s \in \mathbb{R}$ and $\theta \in S^1$. This follows from (50) due to the presence of the factor δ in front of the reparametrization function $h_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}$.

We characterize now the kernel of D_u. We first show that each $\zeta \in \ker D_u$ is a multiple of the (constant) vector corresponding to X_H, or that its component $\zeta^\perp$ on the orthogonal complement vanishes. Let $F(s)$ denote the self-adjoint operator $J_0\partial_\theta + S(s,\theta)$, so that $D_u = \partial_s + F(s)$. If $\zeta \in \ker D_u$ we have $(\partial_s - F(s))(\partial_s + F(s))\zeta = 0$, i.e.

$$\partial_s^2\zeta - F(s)^2\zeta + S'(s)\zeta = 0.$$

By taking the scalar product in $L^2(S^1,\mathbb{R}^{2n})$ with $\zeta^\perp$ and using property (ii) for S we get

$$\langle \zeta^\perp, \partial_s^2\zeta^\perp\rangle - \|F(s)\zeta^\perp\|^2 + \langle \zeta^\perp, S'(s)\zeta^\perp\rangle = 0.$$

The Morse-Bott assumption and property (i) guarantee that $\|F(s)\zeta^\perp\|_{L^2} \ge c\|\zeta^\perp\|_{L^2}$ for some $c > 0$. We obtain

$$\partial_s^2\|\zeta^\perp\|^2_{L^2} \;\ge\; 2\langle\zeta^\perp, \partial_s^2\zeta^\perp\rangle_{L^2} \;\ge\; 2(c^2 - C\delta)\|\zeta^\perp\|^2_{L^2} \;\ge\; c^2\|\zeta^\perp\|^2_{L^2}$$

if $\delta > 0$ is sufficiently small. In particular $\|\zeta^\perp\|^2_{L^2}$ can have no local maximum on $\mathbb{R}$. Since $\|\zeta^\perp\|^2_{L^2} \to 0$ as $s \to \pm\infty$ we deduce that $\zeta^\perp \equiv 0$.

We now show that all elements of $\ker D_u$ are independent of θ. Let $\zeta \in \ker D_u$. Because $\zeta^\perp = 0$ we have $\partial_s\zeta + J_0\partial_\theta\zeta + \delta k(s)\zeta = 0$, with $\partial_s\zeta + \delta k(s)\zeta$ and $\partial_\theta\zeta$ pointwise colinear with X_H. Hence $\partial_s\zeta + \delta k(s)\zeta = 0$ and $\partial_\theta\zeta = 0$.

This shows that the elements of $\ker D_u$ also belong to the kernel of the linearized Morse operator

$$\zeta \mapsto \partial_s\zeta + \delta h'_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}(s) f''_\gamma(\vartheta \circ c(\delta h_{\frac{a}{\delta},\frac{b}{\delta},\frac{\epsilon}{\delta}}))\zeta.$$

This is a differential equation on $\mathbb{R}$ for which the Cauchy problem has a unique solution. Hence the space of solutions is one-dimensional in $C^\infty(\mathbb{R},\mathbb{R})$ and, in order to determine the dimension of $\ker D_u$, we just have to check whether the solutions belong or not to its domain.

If a and b are finite the solutions are constant near $\pm\infty$, hence belong to the domain of D_u and $\dim\ker D_u = 1$. If $a = -\infty$ (and b is finite) we distinguish two cases: either p is a maximum, in which case $f''_\gamma(p) < 0$, the solutions are unbounded near $-\infty$ and $\ker D_u = 0$, or p is a minimum, in which case $f''_\gamma(p) > 0$, the solutions coincide near $-\infty$ with the elements of $\overline{V}'_u$ and $\dim\ker D_u = 1$. Hence $\dim\ker D_u = \mathrm{ind}(p)$. A similar argument shows that $\dim\ker D_u = 1 - \mathrm{ind}(q)$ if $b = +\infty$ (and a is finite). In all cases we have

$$\dim\ker D_u = \mathrm{ind}(D_u),$$

so that D_u is surjective. □

Up to a translation, the defining interval $I(a,b)$ of a gradient cylinder can be considered to be $[-T/2, T/2]$, $T > 0$ in case (i), or $]-\infty, 1]$, $[-1, \infty[$ in cases (ii) and (iii) respectively. We shall thus assume in the sequel that the parameters a, b take the values

$$a = -T/2,\ b = T/2 \ \text{ for } T > 0, \quad \text{ or } a = -\infty,\ b = 1, \quad \text{ or } a = -1,\ b = +\infty.$$

We consider a tuple (γ, a, b, ϵ) and the gradient cylinder $u := u_\delta := u_{\delta,\gamma,a,b,\epsilon}$ for δ small enough. Let (s_δ) be a family of parameters such that $s_\delta \leq s^*_\delta$ and $\frac{s_\delta}{s^*_\delta} \to 1$ as $\delta \to 0$, where

$$s^*_\delta := \begin{cases} (T+\epsilon)/2\delta, & \text{if } a = -T/2,\ b = T/2, \\ 1/\delta, & \text{otherwise.} \end{cases}$$

In particular we have $s_\delta \to \infty$ as $\delta \to 0$. Our goal now is to define modified norms $\|\cdot\|_{1,\delta}$ and $\|\cdot\|_\delta$ on the domain and target of the operators $D_u = D_{u_\delta}$ such that they admit uniformly bounded right inverses with respect to $\delta \to 0$. Let $w_\delta : \mathbb{R} \to \mathbb{R}^+$ be the weight function defined by

$$w_\delta(s) = \begin{cases} e^{d||s|-s_\delta|}, & \text{if } a \text{ and } b \text{ are finite,} \\ e^{d|s-s_\delta|}, & \text{if } a = -\infty \text{ and } b \text{ is finite,} \\ e^{d|s+s_\delta|}, & \text{if } a \text{ is finite and } b = \infty. \end{cases} \tag{55}$$

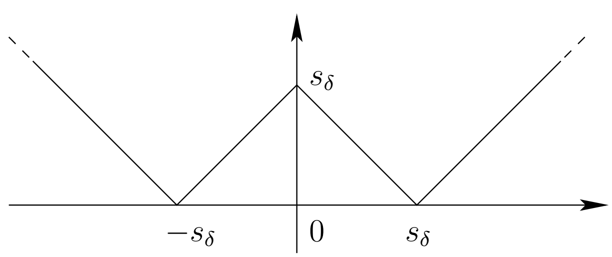

Figure 3: Weight function $||s| - s_\delta|$ for a, b finite (logarithmic scale).

The new norm $\|\cdot\|_\delta$ on the target of D_u is the L^p-norm with weight w_δ, and we emphasize it by writing the target as

$$L^p(\mathbb{R} \times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta).$$

Let $\overline{V}'_{u,\delta}$, $\underline{V}'_{u,\delta}$ be vector spaces of the same dimension as $\overline{V}'_u$, $\underline{V}'_u$, given by (53), and which, when their dimension is nonzero, are spanned by the following sections.

(i) if a, b are both finite the sections are

$$(1 - \beta(s + s_\delta, \theta))X_H(\gamma(\theta + \overline{\theta}_0)) \text{ and } \beta(s - s_\delta, \theta)X_H(\gamma(\theta + \underline{\theta}_0));$$

(ii) if $a=-\infty$ and b is finite the sections are

$$(1-\beta(s-s_\delta,\theta))\nabla f_\gamma(\vartheta\circ u(s,0)) \text{ and } \beta(s-s_\delta,\theta)X_H(\gamma(\theta+\underline{\theta}_0));$$

(iii) if a is finite and $b=+\infty$ the sections are

$$(1-\beta(s+s_\delta,\theta))X_H(\gamma(\theta+\overline{\theta}_0)) \text{ and } \beta(s+s_\delta,\theta)\nabla f_\gamma(\vartheta\circ u(s,0)).$$

In case $a=-\infty$, b finite or a finite, $b=+\infty$ we define the new norm $\|\cdot\|_{1,\delta}$ on the domain of D_u by splitting it as

$$\operatorname{dom} D_u = W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta)\oplus \overline{V}'_{u,\delta}\oplus \underline{V}'_{u,\delta}$$

and setting the norm of the above generators of $\overline{V}'_{u,\delta}$, $\underline{V}'_{u,\delta}$ to be equal to 1.

In case a,b are finite we split the domain of D_u as above and further modify the weighted norm on the $W^{1,p}$-space. We recall from the proof of Proposition 4.8 that $\ker D_u$ is 1-dimensional and is spanned by a section ζ_δ which is constant for $|s|\geq s^*_\delta$. We normalize ζ_δ by requiring that its value at 0 be equal to the constant vector corresponding to X_H. Let $\langle\cdot,\cdot\rangle$ be the scalar product in $L^2(S^1)$. For an element $\zeta\in W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta)$ we denote

$$\kappa_\delta := \frac{\langle\zeta(0,\cdot),\zeta_\delta(0,\cdot)\rangle}{\langle\zeta_\delta(0,\cdot),\zeta_\delta(0,\cdot)\rangle}.$$

We denote

$$\chi_\delta(s,\theta) := \beta(s+s_\delta)\beta(-s+s_\delta)\zeta_\delta(s,\theta)$$

and define the norm $\|\cdot\|_{1,\delta}$ on $W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta)$ by

$$\|\zeta\|_{1,\delta} := \|\zeta-\kappa_\delta\chi_\delta\|_{1,p,\delta}+|\kappa_\delta|.$$

Here $\|\cdot\|_{1,p,\delta}$ is the weighted norm on $W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta)$.

Proposition 4.9. *Let $u=u_\delta=u_{\delta,\gamma,a,b,\epsilon}$ as above. There exists $\delta_2\in]0,\delta_0]$ such that the operator*

$$D_u : (\operatorname{dom} D_u, \|\cdot\|_{1,\delta}) \to (L^p(\mathbb{R}\times S^1, u^*T\widehat{W}; w_\delta(s)dsd\theta), \|\cdot\|_\delta)$$

is surjective and has a uniformly bounded right inverse $Q_u=Q_{u_\delta}$ for $\delta\in]0,\delta_2]$.

Proof. We choose a unitary trivialization of $u^*T\widehat{W}$ as in the proof of Proposition 4.8, so that X_H and $\partial/\partial t$ correspond to constant vectors in $\mathbb{R}^{2n}$, and so that the operator D_u takes the form

$$(D_u\zeta)(s,\theta) = \partial_s\zeta + J_0\partial_\theta\zeta + S(s,\theta)\zeta(s,\theta).$$

Here J_0 is the standard complex structure on $\mathbb{R}^{2n}$ and S is asymptotically symmetric as $s\to\pm\infty$. The matrix $S(s,\theta)$ can be written as $S'(s,\theta)\oplus S''(s,\theta)$ with respect to the splitting $\xi\oplus\langle\partial/\partial t, X_H\rangle$, so that the operator D_u is also

split with respect to the decomposition $\xi \oplus L$, where $L := \langle \partial/\partial t, X_H \rangle$. It is therefore enough to find uniformly bounded right inverses for each of the surjective operators

$$D'_u : W^{1,p}(\mathbb{R} \times S^1, u^*\xi; w_\delta(s) ds d\theta) \to L^p(\mathbb{R} \times S^1, u^*\xi; w_\delta(s) ds d\theta),$$

$$D''_u : W^{1,p}(\mathbb{R} \times S^1, u^*L; \|\cdot\|_{1,\delta}) \oplus \overline{V}'_{u,\delta} \oplus \underline{V}'_{u,\delta} \to L^p(\mathbb{R} \times S^1, u^*L; w_\delta(s) ds d\theta).$$

Here we use the fact that the norm $\|\cdot\|_{1,\delta}$ coincides with the weighted $W^{1,p}$-norm on sections with values in the subbundle $u^*\xi$. Note that D'_u is an isomorphism since it has index 0, whereas $\mathrm{ind}(D''_u) = \mathrm{ind}(D_u)$ is either 0 or 1.

We treat D''_u and consider first the case of a semi-infinite gradient trajectory. The two possible cases are entirely similar, and we assume without loss of generality that $a = -\infty$, $b = 1$. Let

$$S''_0 := \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

so that $\lim_{\delta \to 0} S''(s,\theta) = S''_0$ uniformly in (s,θ). Consider the operator

$$D''_{0,\delta} : W^{1,p}(\mathbb{R} \times S^1, u^*L; w_\delta(s) ds d\theta) \oplus \overline{V}'_{u,\delta} \oplus \underline{V}'_{u,\delta} \to L^p(\mathbb{R} \times S^1, u^*L; w_\delta(s) ds d\theta)$$

defined by $D''_{0,\delta} := \partial_s + J_0 \partial_\theta + S''_0$. As in the proof of Proposition 4.8 one sees that $D''_{0,\delta}$ is surjective, and we claim that it admits a right inverse $Q''_{0,\delta}$ that is uniformly bounded with respect to δ. Indeed, let Q''_0 be a right inverse of $D''_0 := D''_{0,\delta=1}$ and consider the shift operators

$$(T_\delta \zeta)(s) := \zeta(s + s_\delta)$$

acting from $\mathrm{dom}(D''_{0,\delta}) \to \mathrm{dom}(D''_0)$ and from $L^p(w_\delta(s) ds d\theta) \to L^p(e^{d|s|} ds d\theta)$. It follows from the definitions of $\|\cdot\|_{1,\delta}$ and $\|\cdot\|_\delta$ that the operators T_δ are isometries, and we have $D''_{0,\delta} = T_\delta^{-1} D''_0 T_\delta$ since D''_0 is independent of $s \in \mathbb{R}$. Hence $Q''_{0,\delta} = T_\delta^{-1} Q''_0 T_\delta$ is a right inverse for $D''_{0,\delta}$ such that $\|Q''_{0,\delta}\| = \|Q''_0\|$, and the claim is proved.

Now, if δ is small enough we have $\|S''(s,\theta) - S''_0\| \le 1/2\|Q''_0\|$, $s \in \mathbb{R}$ and therefore $\|D''_u - D''_{0,\delta}\| \le 1/2\|Q''_0\|$. This implies that

$$\|D''_u Q''_{0,\delta} - \mathrm{Id}\| = \|D''_u Q''_{0,\delta} - D''_{0,\delta} Q''_{0,\delta}\| \le \frac{1}{2}.$$

Thus $D''_u Q''_{0,\delta}$ is invertible and the norm of its inverse is ≤ 2. Finally a right inverse for D''_u is given by $Q''_{0,\delta}(D''_u Q''_{0,\delta})^{-1}$ and has norm $\le 2\|Q''_0\|$.

We now treat the case $a = -T/2$, $b = T/2$ for $T > 0$. Let $\overline{u} := u_{\delta,\gamma,-(T+\epsilon)/2,0}$, $\underline{u} := u_{\delta,\gamma,0,(T+\epsilon)/2}$ and

$$\overline{D}'' := D''_{\overline{u}} : W^{1,p}(\mathbb{R} \times S^1, \overline{u}^*L; e^{d|s|} ds d\theta) \oplus \overline{V}'_{\overline{u}} \oplus \underline{V}'_{\overline{u}} \to L^p(e^{d|s|} ds d\theta),$$

$$\underline{D}'' := D''_{\underline{u}} : W^{1,p}(\mathbb{R} \times S^1, \underline{u}^*L; e^{d|s|} ds d\theta) \oplus \overline{V}'_{\underline{u}} \oplus \underline{V}'_{\underline{u}} \to L^p(e^{d|s|} ds d\theta).$$

The same argument as above, using the constant operator D''_0, shows that $\overline{D}''$ and $\underline{D}''$ admit right inverses which are uniformly bounded with respect to $\delta \to 0$. Both operators have index 1 and it follows from the description of their kernels given in the proof of Proposition 4.8 that their restrictions to $W^{1,p} \oplus \overline{V}'_u$, respectively $W^{1,p} \oplus \underline{V}'_u$ are isomorphisms. We choose the right inverses $\overline{Q}''$, $\underline{Q}''$ to be the inverses of their respective restrictions.

Let $\underline{\zeta} \in \ker \overline{D}''$, $\overline{\zeta} \in \ker \underline{D}''$ be two sections such that their values at $+\infty$ and respectively $-\infty$ are equal to the (constant) vector corresponding to X_H in the chosen trivialization of $u^*T\widehat{W}$. Let $\underline{V}'$, $\overline{V}'$ be the 1-dimensional vector spaces spanned by $\beta\underline{\zeta}$ and $(1-\beta)\overline{\zeta}$ respectively. Setting the norm of these generators to be equal to 1 defines a new norm on $\mathrm{dom}(\overline{D}'')$ and $\mathrm{dom}(\underline{D}'')$, which we emphasize by decomposing the latter as

$$\mathrm{dom}(\overline{D}'') = W^{1,p}(\mathbb{R} \times S^1, \overline{u}^*L; e^{d|s|}dsd\theta) \oplus \overline{V}'_u \oplus \underline{V}',$$

$$\mathrm{dom}(\underline{D}'') = D''_{\underline{u}} : W^{1,p}(\mathbb{R} \times S^1, \underline{u}^*L; e^{d|s|}dsd\theta) \oplus \overline{V}' \oplus \underline{V}'_u.$$

It follows from our special choice of the right inverses $\overline{Q}''$, $\underline{Q}''$ that the latter are also uniformly bounded with respect to this new norm as $\delta \to 0$.

Let $D'' := \overline{D}'' \#_\delta \underline{D}''$ be the operator obtained by gluing $\overline{D}''$ cut at s_δ and $\underline{D}''$ cut at $-s_\delta$, with the $\|\cdot\|_{1,\delta}$-norm on its domain and the $\|\cdot\|_\delta$-norm on its target. It follows as in [5, Proposition 5] that the right inverses $\overline{Q}''$, $\underline{Q}''$ give rise to a uniformly bounded right inverse Q'' for D'' as $\delta \to 0$. On the other hand, we have $\|D''_u - D''\| \to 0$ as $\delta \to 0$, and we obtain a uniformly bounded right inverse for D''_u by the previous formula $Q''_u := Q''(D''_u Q'')^{-1}$. We note that, upon gluing, the exponential weights at $\pm\infty$ for $\overline{D}''$, $\underline{D}''$ give rise to the peak in the weight function w_δ for D'', and the fibered sum operation on $\underline{V}'$, $\overline{V}'$, on which the norm is fixed, is responsible for the appearance of the distinguished cutoff section ζ_δ leading to the modified norm $\|\cdot\|_{1,\delta}$.

We now treat D'_u and start by making a few general remarks. For each $s_0 \in \mathbb{R}$ the operator

$$D'(s_0) := \partial_s + J_0\partial_\theta + S(s_0,\theta) : W^{1,p}(\mathbb{R}\times S^1, u^*\xi; dsd\theta) \to L^p(\mathbb{R}\times S^1, u^*\xi; dsd\theta)$$

is δ-close to the $\mathbb{R}$-invariant operator with nondegenerate asymptotics corresponding to the constant cylinder over the orbit $u(s_0,\cdot)$. Hence, for $\delta > 0$ small enough, both operators are isomorphisms [24, Lemma 2.4]. Moreover, this property also holds in the presence of weights $e^{d|s|}$, e^{ds} or e^{-ds}. For the weight $e^{d|s|}$ we argue as follows. The operator is still Fredholm between the $W^{1,p}$ and L^p spaces with weights, of the same index 0. Since the corresponding $W^{1,p}$ space is contained in $W^{1,p}(\mathbb{R} \times S^1, u^*\xi; dsd\theta)$ we infer that the operator is injective, hence an isomorphism. For the weight e^{ds} we argue as follows. Multiplication by $e^{\frac{d}{p}s}$ determines linear isomorphisms $M : W^{1,p}(\mathbb{R} \times S^1, u^*\xi; e^{ds}dsd\theta) \to W^{1,p}(\mathbb{R}\times S^1, u^*\xi; dsd\theta)$ and $M : L^p(\mathbb{R}\times S^1, u^*\xi; e^{ds}dsd\theta) \to$

$L^p(\mathbb{R}\times S^1, u^*\xi; dsd\theta)$. The operator $M^{-1}D'(s_0)M$ is an isomorphism and, for $\zeta \in W^{1,p}(\mathbb{R}\times S^1, u^*\xi; e^{ds}dsd\theta)$, we have

$$M^{-1}D'(s_0)M\zeta = D'(s_0)\zeta + \frac{d}{p}\zeta.$$

Since $d>0$ is small as in Proposition A.2 and $p>2$, the operator $M^{-1}D'(s_0)M$ is $\mathbb{R}$-invariant and has nondegenerate asymptotics, hence is an isomorphism [24, Lemma 2.4]. An analogous reasoning using the multiplication by $e^{-\frac{d}{p}s}$ proves the claim for the weight e^{-ds}.

We now prove that D'_u admits a uniformly bounded right inverse in the case $a=-T/2$, $b=T/2$, $s_\delta=(T+\epsilon)/2\delta$. We recall the notation $\overline{u} := u_{\delta,\gamma,-(T+\epsilon)/2,0}$, $\underline{u} := u_{\delta,\gamma,0,(T+\epsilon)/2}$ and set

$$\overline{D}' := D'_{\overline{u}} : W^{1,p}(\mathbb{R}\times S^1, \overline{u}^*\xi; e^{d|s|}dsd\theta) \to L^p(\mathbb{R}\times S^1, \overline{u}^*\xi; e^{d|s|}dsd\theta),$$

$$\underline{D}' := D'_{\underline{u}} : W^{1,p}(\mathbb{R}\times S^1, \underline{u}^*\xi; e^{d|s|}dsd\theta) \to L^p(\mathbb{R}\times S^1, \underline{u}^*\xi; e^{d|s|}dsd\theta).$$

We claim that each of the operators $\overline{D}'$, $\underline{D}'$ is an isomorphism with uniformly bounded right inverse as $\delta\to 0$. We give the proof for $\overline{D}'$ since the proof for $\underline{D}'$ is entirely analogous. We choose a finite number of points $-\infty = s_{-m} < s_{-m+1} < \cdots < s_{-1} < 0 = s_0 < s_1 < \cdots < s_{m+1} = +\infty$ such that $\|S(s,\theta)-S(s',\theta)\| \le 1/4C$ for all $\theta\in S^1$ and $s,s'\in[s_i,s_{i+1}]$, $i=-m,\dots,m$, with $C>0$ a constant to be chosen below. Let

$$b_{i-1} := a_i := c^{-1}(u(s_i,0)), \qquad i=-m,\dots,m+1.$$

We consider the operators

$$\begin{aligned} D'_i &:= D'_{u_{\delta,\gamma,a_{i-1},b_{i-1}}}, \quad i=-m+1,\dots,-1,\\ D'_0 &:= D'_{u_{\delta,\gamma,a_{-1},b_0}},\\ D'_i &:= D'_{u_{\delta,\gamma,a_i,b_i}}, \qquad\quad i=1,\dots,m. \end{aligned}$$

For each $i=-m+1,\dots,m$ we denote by $u_i=u_{i,\delta}$ the gradient cylinder corresponding to the operator D'_i. The domain and range of the operators D'_i are as follows:

$$D'_i : W^{1,p}(\mathbb{R}\times S^1, u_i^*\xi; e^{-ds}dsd\theta) \to L^p(\mathbb{R}\times S^1, u_i^*\xi; e^{-ds}dsd\theta), \quad i<0,$$

$$D'_0 : W^{1,p}(\mathbb{R}\times S^1, u_0^*\xi; e^{d|s|}dsd\theta) \to L^p(\mathbb{R}\times S^1, u_0^*\xi; e^{d|s|}dsd\theta),$$

$$D'_i : W^{1,p}(\mathbb{R}\times S^1, u_i^*\xi; e^{ds}dsd\theta) \to L^p(\mathbb{R}\times S^1, u_i^*\xi; e^{ds}dsd\theta), \quad i>0.$$

We have seen that $D'(s_0)$ is an isomorphism for all $s_0\in\mathbb{R}$ if one uses any of the weights $e^{d|s|}$, e^{ds}, e^{-ds}. Since $S(s_0,\cdot)$ belongs to a compact set of loops of matrices we infer that the norm of the inverse $Q'(s_0) := D'(s_0)^{-1}$ is uniformly bounded with respect to $s_0\in\mathbb{R}$ for each of these three weights. We choose $C := \max_{\mathrm{weight}\in\{e^{d|s|},e^{ds},e^{-ds}\}} \max_{s_0\in\mathbb{R}} \|Q'(s_0)\|$.

The same argument as for D''_u shows that the inverse of each D'_i is bounded by $2C$ independently of δ. We glue together the operators D'_i into $\widetilde{D}'$ by cutting at a_i/δ and b_i/δ. Then $\widetilde{D}'$ is still surjective and the norm of its inverse is bounded by $2C\widetilde{C}^{2m-1}$, with $\widetilde{C}$ a universal constant (see [24, Proposition 3.9]). Note that our choice of weights for the operators D'_i is such that the resulting weight for the domain and target of $\widetilde{D}'$ is still $e^{d|s|}$. On the other hand we have

$$\|\widetilde{D}' - \overline{D}'\| \to 0, \qquad \delta \to 0.$$

This is because the two operators coincide outside $2m-1$ intervals of length 2, where the variation of S tends to zero as $\delta \to 0$. As a consequence the inverse of $\overline{D}'$ is also uniformly bounded when δ is small enough.

We now glue the operator $\overline{D}'$ cut at s_δ with the operator $\underline{D}'$ cut at $-s_\delta$, and denote the resulting operator by D'. The argument in [5, Proposition 5] shows that D' admits a uniformly bounded right inverse Q', provided one uses the weight $w_\delta(s)$ on its domain and target. On the other hand

$$\|D'_u - D'\| \to 0, \qquad \delta \to 0$$

since the two operators differ on a segment of length 2 where the variation of S goes to zero. We infer that D'_u also admits a uniformly bounded right inverse.

The cases when $a = -\infty$, $b = 1$ or $a = -1$, $b = \infty$ follow now easily by combining the proof of the existence of uniformly bounded right inverses for the operators $\overline{D}'$ with the previous use of a shift operator $(T_\delta\zeta)(s) = \zeta(s \pm s_\delta)$. $\square$

Remark 4.10. Note that, if $a = -T/2$, $b = T/2$, Our construction of a right inverse for D'' in the proof of Proposition 4.9 is such that its norm is uniformly bounded as $\delta \to 0$ even if one uses the "non-compensated" norm $\|\cdot\|_{1,p,\delta}$ instead of $\|\cdot\|_{1,\delta}$. However, our choice of the norm $\|\cdot\|_{1,\delta}$ will be essential in the proof of Proposition 4.16.

In order to describe the pregluing construction it is convenient to work with a single section over $\mathcal{B}'_\delta$ rather than with a family of sections. We recall that, up to a translation, the defining interval $I(a,b)$ of a gradient cylinder can be considered to be $[-T/2, T/2]$, $T > 0$ in case (i), or $]-\infty, 1]$, $[-1, \infty[$ in cases (ii) and (iii) respectively. We are therefore led to consider the section

$$\bar{\partial} : \mathcal{B}'_\delta \to \mathcal{E} \tag{56}$$

defined by $\bar{\partial} := \bar{\partial}_{-\infty,1}$ and $\bar{\partial} := \bar{\partial}_{-1,\infty}$ in cases (ii) and (iii), and by

$$\bar{\partial}(u) = \bar{\partial}_\epsilon(u) := \bar{\partial}_{-T_u/2, T_u/2, \epsilon}(u)$$

in case (i). Here $T_u > 0$ is the time needed to flow along the gradient of f_γ from the negative limit to the positive limit of u (see (51)).

Remark 4.11. In case (i) the section $\bar{\partial}$ can be described as follows. The one parameter family of sections $\bar{\partial}_T := \bar{\partial}_{-T/2, T/2, \epsilon}$ gives rise to a section denoted

$\{\bar\partial_T\}$ of the pull-back bundle $\mathrm{pr}_1^*\mathcal{E} \to \mathcal{B}'_\delta \times \mathbb{R}^+$. There is a codimension one embedding $\iota : \mathcal{B}'_\delta \to \mathcal{B}'_\delta \times \mathbb{R}^+$ given by $\iota(u) = (u, T_u)$, the composition $\mathrm{pr}_1 \circ \iota$ is the identity and we have

$$\bar\partial = \{\bar\partial_T\}|_{\mathrm{im}\,\iota}.$$

The situation is summarized in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathrm{pr}_1^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{B}'_\delta \arrow[r, hook, "\iota"] \arrow[u, bend left, "\bar\partial"] & \mathcal{B}'_\delta \times \mathbb{R}^+ \arrow[r, "\mathrm{pr}_1"] \arrow[u, bend left, "\{\bar\partial_T\}"] & \mathcal{B}'_\delta
\end{tikzcd}$$

Given $u \in \mathcal{B}'_\delta$ we denote by $D'_u : T_u\mathcal{B}'_\delta \to \mathcal{E}_u$ the vertical differential of $\bar\partial$. In cases (ii) and (iii) we have seen that D'_u is a Fredholm operator of index $\mathrm{ind}(p)$ and $1 - \mathrm{ind}(q)$ respectively. In case (i) the vertical differential can be computed explicitly as follows. The vertical differential of $\{\bar\partial_T\}$, denoted by $D\{\bar\partial_T\}$, is

$$\begin{aligned}
D\{\bar\partial_T\}(u,T)\cdot(\zeta,\tau) &= D_u^{-T/2,T/2,\epsilon}\zeta - \tau(JX_{H_\delta - H})\frac{d}{dT}h'_{-T/2\delta,T/2\delta,\epsilon/\delta}(s) \\
&= D_u^{-T/2,T/2,\epsilon}\zeta - \frac{\tau}{2\delta}(JX_{H_\delta - H})k_{-T/2\delta,T/2\delta,\epsilon/\delta}(s).
\end{aligned}$$

Let us write a section $\zeta \in T_u\mathcal{B}'_\delta$ as $\zeta = \zeta^0 + a\overline{\zeta} + b\underline{\zeta}$, with $\zeta^0 \in W^{1,p,d}$, $a, b \in \mathbb{R}$ and $\overline{\zeta}$, $\underline{\zeta}$ being the distinguished generators of $\overline{V}'_u$, $\underline{V}'_u$ respectively. The vertical differential D'_u acts by

$$\begin{aligned}
D'_u\zeta &= D\{\bar\partial_T\}(u,T_u)\cdot(\zeta, dT_u\cdot\zeta) \\
&= D_u^{-T_u/2,T_u/2,\epsilon}\zeta - \frac{dT_u\cdot\zeta}{2\delta}(JX_{H_\delta - H})k_{-T_u/2\delta,T_u/2\delta,\epsilon/\delta}(s).
\end{aligned}$$

One can explicitly compute

$$dT_u\cdot\zeta = dT_u\cdot(a\overline{\zeta} + b\underline{\zeta}) = \frac{\dot c(-T_u/2)b - \dot c(T_u/2)a}{\dot c(-T_u/2)\cdot\dot c(T_u/2)},$$

where $c : \mathbb{R} \to S_\gamma$ is the gradient trajectory satisfying $c(-T_u/2) = \overline{\theta}_0$, $c(T_u/2) = \underline{\theta}_0$ and $\dot c$ is the derivative with respect to the X_H-parametrization of S_γ.

Proposition 4.12. *Let $T > 0$ and $u = u_{\delta,\gamma,-T/2,T/2,\epsilon}$. The index of D'_u is equal to 1, its kernel has dimension 2 and a complement of $\mathrm{im}\, D'_u$ is spanned by a section supported in*

$$[-(T+\epsilon)/2\delta, -(T+\epsilon)/2\delta + 1] \times S^1 \quad \bigcup \quad [(T+\epsilon)/2\delta - 1, (T+\epsilon)/2\delta] \times S^1.$$

Morever, D'_u admits a right inverse defined on its image which is uniformly bounded with respect to $\delta \to 0$.

Proof. The first order differential operators D'_u and $D_u^{-T_u/2,T_u/2,\epsilon}$ differ by a term of order zero, hence their indices are equal and $\operatorname{ind} D'_u = 1$.

The operator $D\{\bar\partial_T\}(u, T_u)$ is surjective and has index 2. As a consequence $\dim\ker D'_u \le \dim\ker D\{\bar\partial_T\}(u, T_u) = 2$. Let $c : \mathbb{R} \to S_\gamma$ be the gradient curve defining $u = u_{\delta,\gamma,-T/2,T/2,\epsilon}$. For σ close to zero we define $c^\sigma(s) := c(\sigma + s)$ and denote by $u_1^\sigma := u^\sigma_{\delta,\gamma,-T/2,T/2,\epsilon}$ the gradient cylinder defined by c^σ. Then $\bar\partial(u_1^\sigma) = \bar\partial_T(u_1^\sigma) = 0$, hence $\zeta^1 := \frac{d}{d\sigma}|_{\sigma=0} u_1^\sigma \in \ker D'_u$. We also define $u_2^\sigma := u_{\delta,\gamma,-(T+\sigma)/2,(T+\sigma)/2,\epsilon}$ to be the gradient cylinder associated to c. Then $\bar\partial(u_2^\sigma) = \bar\partial_{T+\sigma}(u_2^\sigma) = 0$, hence $\zeta^2 := \frac{d}{d\sigma}|_{\sigma=0} u_2^\sigma \in \ker D'_u$. Since ζ^1 and ζ^2 are linearly independent, we infer that $\dim\ker D'_u = 2$.

We claim that the section $\eta := \frac{1}{2\delta}(JX_{H_\delta - H})k_{-T_u/2\delta,T_u/2\delta,\epsilon/\delta}(s)$ spans a complement of $\operatorname{im} D'_u$. This follows from (i) in Lemma 4.13 below with $\ell := dT_u$, $\phi := D_u^{-T_u/2,T_u/2,\epsilon}$, $\widetilde\phi := D'_u$, $y := \eta$ and $x_y := \zeta^2$. That D'_u admits a uniformly bounded right inverse defined on its image follows from (ii) in Lemma 4.13 and the fact that $D_u^{-T_u/2,T_u/2,\epsilon}$ has a uniformly bounded right inverse by Proposition 4.9. $\square$

Lemma 4.13. *Let $\phi : E \to F$ be a surjective map of Banach vector spaces, $\ell : E \to \mathbb{R}$ be a nonzero linear functional, $y = \phi(x_y) \in F$ be fixed and $\widetilde\phi : E \to F$ be defined by*

$$\widetilde\phi(x) = \phi(x) - \ell(x)y.$$

We assume that $\ker\phi \subset \ker\ell$. Then $\operatorname{im}\widetilde\phi = \phi(\ker\ell)$ if and only if $\ell(x_y) = 1$, in which case the following hold.

- **(i)** *The element y spans a complement of $\operatorname{im}\widetilde\phi$.*
- **(ii)** *If $Q : F \to E$ is a right inverse for ϕ, then $Q|_{\phi(\ker\ell)}$ is a right inverse for $\widetilde\phi$ defined on its image.*

Proof. We first note that $\operatorname{im}\widetilde\phi \supseteq \phi(\ker\ell)$. Let us now assume that $\operatorname{im}\widetilde\phi = \phi(\ker\ell)$. For $x \notin \ker\ell$ we obtain $\phi(x) - l(x)\phi(x_y) \in \phi(\ker\ell)$, hence $x - l(x)x_y \in \ker\ell$, implying $\ell(x) - \ell(x)\ell(x_y) = 0$ and $\ell(x_y) = 1$. Conversely, if $\ell(x_y) = 1$ we obtain $x - \ell(x)x_y \in \ker\ell$ for any $x \in E$, hence $\widetilde\phi(x) = \phi(x - \ell(x)x_y) \in \phi(\ker\ell)$.

The element y does not belong to $\phi(\ker\ell)$ because $y = \phi(x_y)$ with $\ell(x_y) = 1$ and the preimage x_y is well-defined up to an element of $\ker\phi \subset \ker\ell$. This proves the equivalence in the statement of the Lemma, as well as (i).

To prove (ii) we need to show that $Q(\phi(\ker\ell)) \subset \ker\ell$. We prove the stronger statement $\operatorname{im} Q \cap \ker\ell = Q(\phi(\ker\ell))$. The inclusion $\operatorname{im} Q \cap \ker\ell \subset Q(\phi(\ker\ell))$ follows from the observation that, given $x = Qz$ with $\ell(x) = 0$, we have $z = \phi(Qz) = \phi(x) \in \phi(\ker\ell)$. On the other hand note that $Q\phi$ is the projection to $\operatorname{im} Q$ along $\ker\phi$. Since $\ker\phi \subset \ker\ell$, it follows that $Q\phi(\ker\ell) \subset \operatorname{im} Q \cap \ker\ell$. $\square$

We describe now the pre-gluing construction for elements of the Morse-Bott moduli spaces and gradient cylinders of the form $u_{\delta,\gamma,a,b,\epsilon}$. We define the space

$$\widetilde{\mathcal{B}}_\delta := \widetilde{\mathcal{B}}_\delta^{1,p,d}(\overline{\gamma}_p, S_{\gamma_{m-1}}, \ldots, S_{\gamma_1}, \underline{\gamma}_q, A; H, \{f_\gamma\})$$

consisting of tuples $\widetilde{w} := (u_1, \dots, u_m, v_0, \dots, v_m)$ satisfying the following conditions.

(i) $u_i \in \mathcal{B}^{1,p,d}(S_{\gamma_i}, S_{\gamma_{i-1}}, A_i; H)$, $i = 1, \dots, m$, with $S_{\gamma_0} := S_{\underline{\gamma}}$, $S_{\gamma_m} := S_{\overline{\gamma}}$, $S_{\gamma_i} \neq S_{\gamma_{i-1}}$, $i = 1, \dots, m$ and $A_1 + \dots + A_m = A$;

(ii) $v_0 \in \mathcal{B}^{1,p,d}_\delta(S_{\underline{\gamma}}, q; f_{\underline{\gamma}})$, $v_i \in \mathcal{B}^{1,p,d}_\delta(S_{\gamma_i}, S_{\gamma_i}; f_{\gamma_i})$ for $i = 1, \dots, m-1$, and $v_m \in \mathcal{B}^{1,p,d}_\delta(p, S_{\overline{\gamma}}; f_{\overline{\gamma}})$;

(iii) $\overline{\mathrm{ev}}(v_{i-1}) = \underline{\mathrm{ev}}(u_i)$ and $\underline{\mathrm{ev}}(v_i) = \overline{\mathrm{ev}}(u_i)$ for $i = 1, \dots, m$;

(iv) $\overline{\mathrm{ev}}(v_0)$ belongs to the stable manifold of q, and $\underline{\mathrm{ev}}(v_m)$ belongs to the unstable manifold of p.

By the definition of the spaces $\mathcal{B}^{1,p,d}_\delta(S_{\gamma_i}, S_{\gamma_i}; f_{\gamma_i})$ we have $\overline{\mathrm{ev}}(v_i) \neq \underline{\mathrm{ev}}(v_i)$ for $i = 1, \dots, m-1$. We denote by $T_i > 0$ the unique positive real number such that $\varphi^{f_{\gamma_i}}_{T_i}(\overline{\mathrm{ev}}(v_i)) = \underline{\mathrm{ev}}(v_i)$, where $\varphi^{f_\gamma}_s$ is the gradient flow of f_γ.

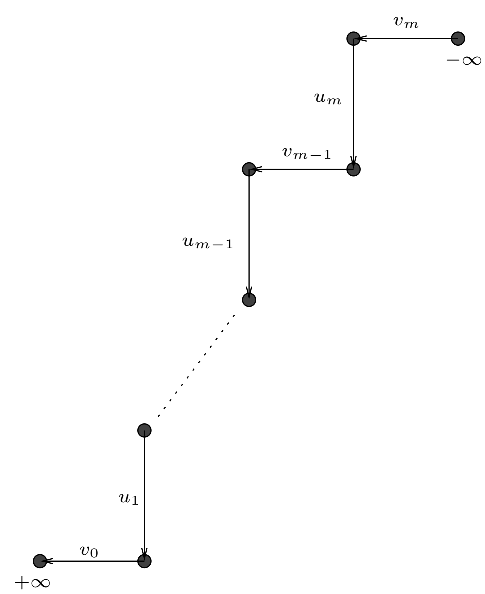

Figure 4: Broken Morse-Bott trajectory $\widetilde{w}$.

Let us choose a tubular neighbourhood $U_\gamma \subset \widehat{W}$ for each $\gamma \in \mathcal{P}(H)$, parametrized by $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$. Given any subset

$$\mathcal{K} \subset \widetilde{\mathcal{B}}^{1,p,d}_\delta(\overline{\gamma}_p, S_{\gamma_{m-1}}, \dots, S_{\gamma_1}, \underline{\gamma}_q, A; H, \{f_\gamma\})$$

for which there exists $s_0 > 0$ such that, for $|s| \geq s_0$, the components of any $\widetilde{w} \in \mathcal{K}$ belong to the respective tubular neighbourhoods of their asymptotics,

we construct, for $\delta > 0$ small enough and $\epsilon_i \in \mathbb{R}$, $i = 1, \dots, m-1$ small enough in absolute value, a pre-gluing map

$$G_{\delta,\overline{\epsilon}} : \mathcal{K} \to \mathcal{B}^{1,p,d}_{\delta}(\overline{\gamma}_p, \underline{\gamma}_q, A; H, \{f_\gamma\}), \quad \overline{\epsilon} := (\epsilon_1, \dots, \epsilon_{m-1}).$$

Let $\beta : \mathbb{R} \to [0,1]$ be a smooth increasing cutoff function vanishing on $]-\infty, 0]$ and identically equal to 1 on $[1, \infty[$. Define the gluing profile $R = R(\delta)$ by

$$R := \frac{1}{2d} \ln\big(\frac{1}{\delta}\big). \tag{57}$$

We define for $i = 1, \dots, m$ the maps $\widehat{u}_i : [-R, R] \times S^1 \to \widehat{W}$ by

$$\widehat{u}_i(s,\theta) := \begin{cases} \begin{cases} z(s,\theta) = \beta(s+R) z \circ u_i(s,\theta), \\ \vartheta(s,\theta) = \theta + \beta(s+R)(\vartheta \circ u_i(s,\theta) - \theta), \end{cases} & s \in [-R, -R+1], \\ u_i(s,\theta), & s \in [-R+1, R-1], \\ \begin{cases} z(s,\theta) = \beta(-s+R) z \circ u_i(s,\theta), \\ \vartheta(s,\theta) = \theta + \beta(-s+R)(\vartheta \circ u_i(s,\theta) - \theta), \end{cases} & s \in [R-1, R]. \end{cases}$$

We define for $i = 1, \dots, m-1$ the maps

$$\widehat{v}_i : [-(T_i + \epsilon_i)/2\delta, (T_i + \epsilon_i)/2\delta] \times S^1 \to \widehat{W}$$

by the analogous formulas in which we replace R by $\frac{T_i + \epsilon_i}{2\delta}$. We also define

$$\widehat{v}_0 : [-1/\delta, +\infty[\times S^1 \to \widehat{W}$$

by

$$\widehat{v}_0(s,\theta) := \begin{cases} \begin{cases} z(s,\theta) = \beta(s + \frac{1}{\delta}) z \circ v_0(s,\theta), \\ \vartheta(s,\theta) = \theta + \beta(s + \frac{1}{\delta})(\vartheta \circ v_0(s,\theta) - \theta), \end{cases} & s \in [-\frac{1}{\delta}, -\frac{1}{\delta} + 1], \\ v_0(s,\theta), & s \in [-\frac{1}{\delta} + 1, +\infty[, \end{cases}$$

as well as

$$\widehat{v}_m :]-\infty, 1/\delta] \times S^1 \to \widehat{W}$$

by the analogous formula with s replaced by $-s$ and v_0 replaced by v_m. Finally, we define

$$G_{\delta,\overline{\epsilon}}(\widetilde{w})$$

as the catenation $\widehat{v}_m, \widehat{u}_m, \widehat{v}_{m-1}, \dots, \widehat{u}_1, \widehat{v}_0$. The catenation of these maps is performed in the above order and with (obvious) shifts

$$0 = s_{v_m} < s_{u_m} < s_{v_{m-1}} < \dots < s_{u_1} < s_{v_0}$$

in the domain defined by

$$\begin{aligned} s_{u_j} &= s_{v_j} + \ell_j, \\ s_{v_{j-1}} &= s_{u_j} + \ell_{j-1} \end{aligned} \tag{58}$$

for $j = 1, \dots, m$. Here we denote

$$\ell_i := R + (T_i + \varepsilon_i)/2\delta \tag{59}$$

for $i = 0, \dots, m$, with the convention $T_m = T_0 = 2$ and $\varepsilon_m = \varepsilon_0 = 0$. We have in particular

$$\widehat{v}_i(s,\theta) = G_{\delta,\overline{\epsilon}}(\widetilde{w})(s + s_{v_i}, \theta), \quad (s,\theta) \in \mathrm{dom}(\widehat{v}_i), \quad i = 0, \dots, m,$$
$$\widehat{u}_j(s,\theta) = G_{\delta,\overline{\epsilon}}(\widetilde{w})(s + s_{u_j}, \theta), \quad (s,\theta) \in \mathrm{dom}(\widehat{u}_j), \quad j = 1, \dots, m.$$

Given $\mathbf{u} = (c_m, u_m, \dots, u_1, c_0) \in \widehat{\mathcal{M}}^A(p,q;H,\{f_\gamma\},J)$, we denote by

$$G_{\delta,\overline{\epsilon}}(\mathbf{u})$$

the element $G_{\delta,\overline{\epsilon}}(\widetilde{w}) \in \mathcal{B}_\delta$, where $\widetilde{w} := (v_m, u_m, \dots, u_1, v_0)$ and $v_i := u_{\delta,\gamma_i,a_i,b_i,\epsilon_i}$, $i = 0, \dots, m$ is the gradient cylinder corresponding to the gradient trajectory $c_i : I(a_i,b_i) \to S_{\gamma_i}$.

The section $\overline{\partial}_{H_\delta,J}(G_{\delta,\overline{\epsilon}}(\widetilde{w}))$ belongs to the space

$$L^p(\mathbb{R} \times S^1, G_{\delta,\overline{\epsilon}}(\widetilde{w})^* T\widehat{W}; g_{\delta,\overline{\epsilon}}(s) ds d\theta),$$

where the continuous function $g_{\delta,\overline{\epsilon}}(s)$ is the catenation of the following functions:

(i) $g_{\delta,u_i}(s) := e^{d|s|}$ on the domain $[-R,R]$ of $\widehat{u}_i$;

(ii) $g_{\delta,\epsilon_i,v_i}(s) = e^{d||s|-s_{i,\delta}|}$ on the domain $[-(T_i+\epsilon_i)/2\delta, (T_i+\epsilon_i)/2\delta]$ of $\widehat{v}_i$, where $s_{i,\delta} = \frac{T_i+\epsilon_i}{2\delta} - R \le s^*_{i,\delta} = \frac{T_i+\epsilon_i}{2\delta}$, $i = 1, \dots, m-1$;

(iii) $g_{\delta,v_0}(s) := e^{d|s+s_{0,\delta}|}$ on the domain $[-1/\delta, +\infty[$ of $\widehat{v}_0$, where $s_{0,\delta} = 1/\delta - R \le s^*_{0,\delta} = 1/\delta$;

(iv) $g_{\delta,v_m}(s) := e^{d|s-s_{m,\delta}|}$ on the domain $]-\infty, 1/\delta]$ of $\widehat{v}_m$, with $s_{m,\delta} = 1/\delta - R \le s^*_{m,\delta} = 1/\delta$.

We denote the norm on the above L^p space with weight $g_{\delta,\overline{\epsilon}}$ by $\|\cdot\|_\delta$, omitting in the notation the dependence on the numbers $T_i + \epsilon_i$, $i = 1, \dots, m-1$. We define a norm $\|\cdot\|_{1,\delta}$ on the space

$$W^{1,p}(\mathbb{R} \times S^1, G_{\delta,\overline{\epsilon}}(\widetilde{w})^* T\widehat{W}; g_{\delta,\overline{\epsilon}}(s) ds d\theta)$$

as follows. For $j = 1 \dots, m$ let

$$\overline{\kappa}_j = \frac{\langle \zeta(s_{u_j} - R, \cdot), X_H \rangle}{\langle X_H, X_H \rangle}, \qquad \underline{\kappa}_j = \frac{\langle \zeta(s_{u_j} + R, \cdot), X_H \rangle}{\langle X_H, X_H \rangle}, \tag{60}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(S^1)$. Here $s_{u_j} - R$ and $s_{u_j} + R$ are the coordinates of the catenation circles between $\widehat{u}_j$ and $\widehat{v}_j$, respectively $\widehat{u}_j$ and $\widehat{v}_{j-1}$. For $i = 1, \dots, m-1$ let

$$\kappa_i = \frac{\langle \zeta(s_{v_i}, \cdot), \zeta_{i,\delta}(0,\cdot) \rangle}{\langle \zeta_{i,\delta}(0,\cdot), \zeta_{i,\delta}(0,\cdot) \rangle},$$

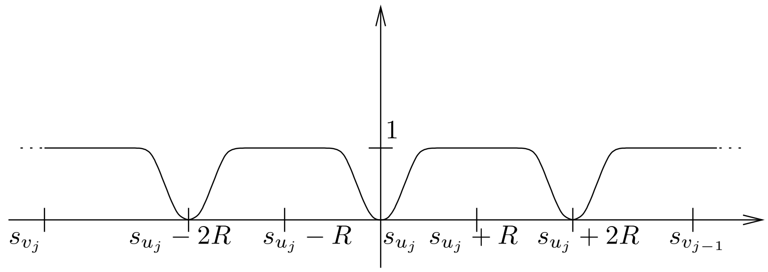

Figure 5: The definition of $\|\cdot\|_{1,\delta}$.

where the section $\zeta_{i,\delta}$ generates the kernel of the operator D_{v_i} as in Proposition 4.8. The norm $\|\cdot\|_{1,\delta}$ is then defined by

$$\begin{aligned}\|\zeta\|_{1,\delta} :=& \\ &\Big\|\zeta - \sum_{j=1}^{m} \overline{\kappa}_j \beta(-s+s_{u_j})\beta(s-s_{u_j}+2R)X_H \\ &\qquad - \underline{\kappa}_j \beta(s-s_{u_j})\beta(-s+s_{u_j}+2R)X_H \\ &- \sum_{i=1}^{m-1} \kappa_i \beta(s-s_{v_i}+\ell_i-2R)\beta(-s+s_{v_i}+\ell_i-2R)\zeta_{i,\delta}(\cdot - s_{v_i},\cdot)\Big\|_{W^{1,p}(g_{\delta,\overline{\epsilon}})} \\ &+ \sum_{j=1}^{m}\big(|\overline{\kappa}_j| + |\underline{\kappa}_j|\big) + \sum_{i=1}^{m-1}|\kappa_i|.\end{aligned} \tag{61}$$

Here ℓ_j is defined by (59), $\beta : \mathbb{R} \to [0,1]$ is the smooth cutoff function which vanishes on $]-\infty,0]$ and is equal to 1 on $[1,\infty[$, and $\|\cdot\|_{W^{1,p}(g_{\delta,\overline{\epsilon}})}$ is the $W^{1,p}$-norm with weight $g_{\delta,\overline{\epsilon}}$ on $W^{1,p}(e^{d|s|}dsd\theta)$. The graph of the function

$$\begin{aligned}&\beta(-s+s_{u_j})\beta(s-s_{u_j}+2R) + \beta(s-s_{u_j})\beta(-s+s_{u_j}+2R) \\ &\quad +\beta(s-s_{v_j}+\ell_j-2R)\beta(-s+s_{v_j}+\ell_j-2R) \\ &\quad +\beta(s-s_{v_{j-1}}+\ell_{j-1}-2R)\beta(-s+s_{v_{j-1}}+\ell_{j-1}-2R)\end{aligned}$$

is depicted in Figure 5.

Remark 4.14. The definition of $\|\cdot\|_{1,\delta}$ is such that the norm of the gluing map G constructed in the proof of Proposition 4.18 below is uniformly bounded with respect to $\delta \to 0$.

Proposition 4.15. *Let $\widetilde{w} \in \widetilde{\mathcal{B}}_\delta$ and $\overline{\epsilon}(\delta) := (\epsilon_1(\delta),\dots,\epsilon_{m-1}(\delta))$ be such that*

(i) $\epsilon_i(\delta) \to 0$, $\delta \to 0$ *for* $i = 1,\dots,m-1$*;*

(ii) $u_i \in \mathcal{M}^{A_i}(S_{\gamma_i}, S_{\gamma_{i-1}}; H, J)$, $i = 1,\dots,m$*;*

(iii) *the components v_i are of the form $u_{\delta,\gamma_i,a_i,b_i,\epsilon_i}$, with $b_i = -a_i = T_i/2$ for $i = 1,\dots,m-1$, $b_0 = +\infty$, $a_0 = -1$, $\epsilon_0 = 0$ and $b_m = 1$, $a_m = -\infty$, $\epsilon_m = 0$.*

Then

$$\lim_{\delta\to 0} \|\bar\partial_{H_\delta,J}(G_{\delta,\overline{\epsilon}}(\widetilde{w}))\|_\delta = 0.$$

Proof. We must check that $\|\bar\partial_{H_\delta,J}(G_{\delta,\overline{\epsilon}}(\widetilde{w}))|_{I\times S^1}\|_\delta \to 0$ as $\delta\to 0$ when $I\subset\mathbb{R}$ is an interval of the following type.

(i) $I = [-R+1, R-1]$ is contained in the domain of $\widehat{u}_i$. Then $\bar\partial_{H_\delta,J}(\widehat{u}_i) = -J(X_{H_\delta} - X_H)\circ\widehat{u}_i$. The norm of this map is pointwise bounded by a constant multiple of δ. Hence its δ-norm is bounded by a constant multiple of $\delta e^{dR}\to 0$, $\delta\to 0$;

(ii) $I = [-R, -R+1]$ or $I = [R-1, R]$ is contained in the domain of $\widehat{u}_i$. We have $\bar\partial_{H_\delta,J}(\widehat{u}_i) = \bar\partial_{H,J}(\widehat{u}_i) - J(X_{H_\delta} - X_H)\circ\widehat{u}_i$. The second term is bounded as in (i). The term $\bar\partial_{H,J}(\widehat{u}_i)$ is pointwise bounded by the norms of $z\circ\widehat{u}_i$, $\vartheta\circ\widehat{u}_i - \theta$ and of their derivatives. By Proposition A.1 their δ-norm is bounded by a constant multiple of $e^{(d-r)R}\to 0$, $\delta\to 0$;

(iii) $I = [-(T_i+\epsilon_i)/2\delta + 1, (T_i+\epsilon_i)/2\delta - 1]$ for $i = 1,\dots m-1$, or $I = [-1/\delta + 1, +\infty[$ or $I =]-\infty, 1/\delta - 1]$ and is contained in the domain of some $\widehat{v}_i$. Since $\bar\partial_{H'_{-T_i/2,T_i/2,\epsilon_i},J}(\widehat{v}_i) = 0$ and $H'_{-T_i/2,T_i/2,\epsilon_i} = H_\delta$ for $s\in I$, we already have $\|\bar\partial_{H_\delta,J}(G_{\delta,\overline{\epsilon}}(\widetilde{w}))|_{I\times S^1}\|_\delta = 0$;

(iv) $I = [-(T_i+\epsilon_i)/2\delta, -(T_i+\epsilon_i)/2\delta + 1]$ or $I = [(T_i+\epsilon_i)/2\delta - 1, (T_i+\epsilon_i)/2\delta]$ for $i = 1,\dots m-1$, or $I = [-1/\delta, -1/\delta + 1]$, or $I = [1/\delta - 1, 1/\delta]$ and is contained in the domain of some $\widehat{v}_i$. Then $\bar\partial_{H_\delta,J}(\widehat{v}_i)$ involves only $\vartheta\circ\widehat{v}_i - \theta$, its derivative with respect to s and $\delta\nabla f_{\gamma_i}$. By formula (50) the norm of these expressions is pointwise bounded by a constant multiple of δ, therefore their δ-norms are bounded by $\delta e^{dR}\to 0$ as $\delta\to 0$.

□

Proposition 4.16. *Let $[\widetilde{v}_n]\in\mathcal{M}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$ with $\delta_n\to 0$, $n\to\infty$ and let $[\mathbf{u}]\in\mathcal{M}^A(p,q;H,\{f_\gamma\},J)$ be a broken Floer trajectory of level $\ell = 1$ whose intermediate gradient fragments $c_1,\dots,c_{m-1}$ are nonconstant. Then $[\widetilde{v}_n]\to[\mathbf{u}]$, $n\to\infty$ if and only if there exist*

- *representatives $v_n\in[\widetilde{v}_n]$, $\mathbf{v}\in[\mathbf{u}]$,*
- *real parameters $\overline{\epsilon}^n = (\epsilon_1^n,\dots,\epsilon_{m-1}^n)$ with $\epsilon_i^n\to 0$, $n\to\infty$,*
- *vector fields $\zeta_n\in T_{G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})}\mathcal{B}_\delta$ with $\zeta_n = (\zeta_n^0, \overline{\zeta}_n, \underline{\zeta}_n)$, such that*

$$\|\zeta_n\|_{1,\delta_n} := \|\zeta_n^0\|_{1,\delta_n} + \|\overline{\zeta}_n\| + \|\underline{\zeta}_n\| \to 0, \qquad n\to\infty,$$

satisfying

$$v_n := \exp_{G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})}(\zeta_n).$$

Proof. We first prove the converse implication, namely that convergence in norm implies geometric convergence. We define shifts (s_i^n), $i=1,\ldots,m$ inductively by

$$s_m^n := 1/\delta_n + R_n, \qquad s_i^n := s_{i+1}^n + 2R_n + (T_i + \epsilon_i^n)/\delta_n.$$

We claim that $v_n(\cdot + s_i^n, \cdot) \to u_i$, $n\to\infty$ uniformly on compact sets. Let $R_0 > 0$ be fixed. By assumption

$$\|\zeta_n^0(\cdot + s_i^n, \cdot)|_{[-R_0,R_0]\times S^1}\|_{1,\delta_n} \to 0, \quad n\to\infty.$$

By the Sobolev embedding theorem this implies

$$\|\zeta_n^0(\cdot + s_i^n, \cdot)|_{[-R_0,R_0]\times S^1}\|_{C^0} \to 0, \quad n\to\infty.$$

Since

$$G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})(\cdot + s_i^n, \cdot)|_{[-R_0,R_0]\times S^1} = u_i|_{[-R_0,R_0]\times S^1}$$

for n sufficiently large, the conclusion follows.

We now prove the direct implication. Let us pick a representative

$$\mathbf{v} = (c_m, u_m, c_{m-1}, \ldots, u_1, c_0) \in [\mathbf{u}]$$

and let T_i, $i=0,\ldots,m$ be the lengths of the intervals of definition of c_i, with the convention $T_0 = T_m = +\infty$. We also choose arbitrary representatives $v_n \in [\widetilde{v}_n]$. By assumption there exist shifts (s_i^n) such that $v_n(\cdot + s_i^n, \cdot)$ converges to u_i uniformly on compact sets. We define

$$\epsilon_i^n := \delta_n(s_i^n - s_{i+1}^n - 2R_n) - T_i, \qquad i=1,\ldots,m-1. \tag{62}$$

By Lemma 4.6 we have $\epsilon_i^n \to 0$, $n\to\infty$. We define partitions of the real line

$$-\infty = a_m^n \le b_m^n \le a_{m-1}^n \le \ldots \le a_0^n \le b_0^n = +\infty$$

by $b_m^n := 1/\delta_n$ and

$$a_{i-1}^n := b_i^n + 2R_n, \quad b_{i-1}^n := a_{i-1}^n + (T_{i-1} + \epsilon_{i-1}^n)/\delta_n, \quad i=1,\ldots,m.$$

We define a sequence of shifts (s^n) by

$$s^n := s_m^n - 1/\delta_n - R_n$$

and we still denote by v_n the shifted sequence $v_n(\cdot + s^n, \cdot)$.

We first show the existence of a unique vector field ζ_n satisfying $v_n = \exp_{G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})}(\zeta_n)$. For that it is enough to prove

$$\lim_{n\to\infty} \sup_{s\in I_n, \theta\in S^1} \mathrm{dist}(v_n(s,\theta), G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})(s,\theta)) = 0, \tag{63}$$

where I_n is an interval of the following form:

(i) $[b_i^n, a_{i-1}^n]$, $i = 1, \dots, m$;

(ii) $[a_i^n, b_i^n]$, $i = 1, \dots, m-1$;

(iii) $[b_m^n - K/\delta_n, b_m^n]$ or $[a_0^n, a_0^n + K/\delta_n]$, for any $K > 0$.

The asymptotic behaviour of v_n and $G_{\delta_n, \overline{\epsilon}^n}(\mathbf{v})$ ensures that ζ_n is an element of the relevant $W^{1,p}$-space.

We prove case (i) by contradiction. Assume that there exists $\epsilon > 0$ and a sequence $(\widetilde{s}_n, \widetilde{\theta}_n) \in [b_i^m, a_{i-1}^n] \times S^1$ such that

$$\mathrm{dist}(v_n(\widetilde{s}_n, \widetilde{\theta}_n), G_{\delta_n, \overline{\epsilon}^n}(\mathbf{v})(\widetilde{s}_n, \widetilde{\theta}_n)) \geq \epsilon.$$

Since (63) is satisfied if one replaces v_n by $u_i(\cdot - s_i^n, \cdot)$ (by definition of $G_{\delta_n, \overline{\epsilon}^n}(\mathbf{v})$), we also have

$$\mathrm{dist}(v_n(\widetilde{s}_n, \widetilde{\theta}_n), u_i(\widetilde{s}_n - s_i^n, \widetilde{\theta}_n)) \geq \epsilon/2 \tag{64}$$

for n large enough. By the assumption of uniform convergence on compact sets $v_n(\cdot + s_i^n, \cdot) \to u_i(\cdot, \cdot)$, up to passing to a subsequence we can assume that $\widetilde{s}_n - s_i^n \to \pm\infty$. We treat the case $\widetilde{s}_n - s_i^n \to \infty$, the other case being similar. Since $\widetilde{s}_n \in [b_i^n, a_{i-1}^n]$ and $\delta_n(a_{i-1}^n - b_i^n) = 2\delta_n R_n \to 0$, we have $\delta_n(\widetilde{s}_n - s_i^n) \to 0$. By Lemma 4.5 we infer that $v_n(\cdot + \widetilde{s}_n, \cdot) \to \underline{\mathrm{ev}}(u_i)$, which means

$$\lim_{n \to \infty} v_n(\widetilde{s}_n, \cdot) = \lim_{n \to \infty} u_i(\widetilde{s}_n - s_i^n, \cdot)$$

and this contradicts (64).

Note that the above proof shows that $v_n(\cdot + a_{i-1}^n, \cdot) \to \underline{\mathrm{ev}}(u_i)$ and $v_n(\cdot + b_i^n, \cdot) \to \overline{\mathrm{ev}}(u_i)$, $i = 1, \dots, m$ uniformly on compact sets.

We now prove case (ii). Let us fix $1 \leq i \leq m-1$. An action argument as the one in the proof of Lemma 4.6 shows that $v_n(I_n \times S^1)$ is entirely contained in a small neighbourhood of S_{γ_i}. We apply Proposition A.3 to v_n and $I_n \times S^1$ to obtain

$$\lim_{n \to \infty} \sup_{(s,\theta) \in I_n \times S^1} |z \circ v_n(s, \theta)| = 0$$

and

$$\lim_{n \to \infty} \sup_{(s,\theta) \in I_n \times S^1} |\vartheta \circ v_n(s, \theta) - \theta - \varphi_{\delta_n(s - a_i^n)}^{f_{\gamma_i}}(\underline{\mathrm{ev}}(u_{i+1}))| = 0.$$

The same two equations hold, by definition, if one replaces v_n by $G_{\delta_n, \overline{\epsilon}^n}(\mathbf{v})$, and the conclusion follows.

We now prove (iii). We treat only the case $I_n = [a_0^n, a_0^n + K/\delta_n]$, the other case being similar. An action argument as above shows that $v_n(\cdot + a_0^n + K/\delta_n, \cdot)$ converges uniformly on compact sets to a constant cylinder over some orbit $\gamma \in S_{\gamma_0}$. By Lemma 4.6 we know that $\gamma = \varphi_K^{f_{\gamma_0}}(\underline{\mathrm{ev}}(u_1))$, and in particular is not a critical point of f_{γ_0}. Now the conclusion follows in the same way as in case (ii).

We now show that

$$\lim_{n \to \infty} \|\zeta_n|_{I_n \times S^1}\|_{1, \delta_n} = 0 \tag{65}$$

in each of the cases (i)-(iii). We denote in the sequel

$$|\zeta(s,\theta)|_1 := |\zeta(s,\theta)| + |\nabla_s\zeta(s,\theta)| + |\nabla_\theta\zeta(s,\theta)|.$$

We first consider case (i). Let us fix $K>0$ large enough. For n large enough we can write

$$I_n = [s_i^n - R_n, s_i^n - K] \cup [s_i^n - K, s_i^n + K] \cup [s_i^n + K, s_i^n + R_n].$$

We first note that

$$\begin{aligned}\int_{s_i^n-K}^{s_i^n+K} |\zeta(s,\theta)|_1^p \; g_{\delta_n,\overline{\epsilon}^n}(s) ds d\theta &= \int_{s_i^n-K}^{s_i^n+K} |\zeta(s,\theta)|_1^p \; e^{d|s-s_i^n|} ds d\theta \\ &\le \sup_{\substack{\theta\in S^1 \\ s\in[s_i^n-K,s_i^n+K]}} |\zeta(s,\theta)|_1^p \cdot e^{dK}.\end{aligned}$$

Since $v_n(\cdot + s_i^n, \cdot)$ and $G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})(\cdot + s_i^n,\cdot)$ converge uniformly on compact sets together with their derivatives to u_i, the last term goes to zero as $n\to\infty$.

In order to estimate the integral on the interval $[s_i^n - R_n, s_i^n - K]$ we apply Proposition A.3 on $[s_{i+1}^n + K, s_i^n - K]$ to v_n to obtain

$$|z\circ v_n(s,\theta)|_1 \le C(K)\frac{\cosh(\rho(s-\frac{s_{i+1}^n+s_i^n}{2}))}{\cosh(\rho(\frac{s_i^n-s_{i+1}^n}{2}-K))} \le C_1C(K)e^{\rho(s-s_i^n+K)}$$

and

$$|\vartheta\circ v_n(s,\theta) - \theta - \varphi_{\delta_n(s-b_i^n)}^{f_{\gamma_i}}(p_i^n)|_1 \le C_1C(K)e^{\rho(s-s_i^n+K)},$$

where $|\cdot|_1$ stands for the pointwise C^1-norm, for some $p_i^n\in S_{\gamma_i}$ such that $p_i^n\to\overline{\mathrm{ev}}(u_i)$, $n\to\infty$. Similar estimates hold, by definition, if one replaces v_n by $G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v})$ and p_i^n with $\overline{\mathrm{ev}}(u_i)$. Hence we obtain

$$|\zeta_n(s,\theta) - \overline{\kappa}_i^n X_H|_1 \le C_1C(K)e^{\rho(s-s_i^n+K)}, \tag{66}$$

where $\overline{\kappa}_i^n\to 0$ as $n\to\infty$ and

$$\begin{aligned}C(K) = C\max(&\|Q_\infty v_n(s_{i+1}^n+K)\|, \|Q_\infty v_n(s_i^n-K)\|, \\ &\|Q_\infty\widetilde{v}_n(s_{i+1}^n+s_n+K)\|, \|Q_\infty\widetilde{v}_n(s_i^n+s_n-K)\|).\end{aligned} \tag{67}$$

We obtain

$$\begin{aligned}&\int_{s_i^n-R_n}^{s_i^n-K} |\zeta_n(s,\theta) - \overline{\kappa}_i^n X_H|_1^p \; g_{\delta_n}(s) ds d\theta \\ &= \int_{s_i^n-R_n}^{s_i^n-K} |\zeta_n(s,\theta) - \overline{\kappa}_i^n X_H|_1^p \; e^{-d(s-s_i^n)} ds d\theta \\ &\le C_2 C(K)^p e^{dK}.\end{aligned}$$

A similar estimate holds when the interval of integration is $[s_i^n + K, s_i^n + R_n]$, with $C(K)$ replaced with $C'(K)$. Letting $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{I_n\times S^1} & |\zeta_n(s,\theta) - \overline{\kappa}_i^n\beta(-s+s_i^n)X_H - \underline{\kappa}_i^n\beta(s-s_i^n)X_H|_1^p \; g_{\delta_n}(s)dsd\theta \\ \leq \quad & C_2(C(K)^p + C'(K)^p)e^{dK}.\end{aligned}$$

We let now $K \to \infty$. Proposition A.3 implies that, for $K > K'$, we have $C(K') \le C_3 C(K)e^{-\rho(K'-K)}$, hence $C(K)^p e^{dK} \to 0$ as $K \to \infty$ because $d < \rho p$. The equality (65) follows.

We now consider case (ii). We fix $K > 0$ large enough and apply Proposition A.3 on the interval $[s_{i+1}^n + K, s_i^n - K] \supset [s_{i+1}^n + R_n, s_i^n - R_n] = I_n$ to obtain as in case (i)

$$|\zeta_n(s,\theta) - \kappa_i^n\zeta_{i,\delta}(s,\theta)|_1 \le C(K)\frac{\cosh(\rho(s - \frac{s_{i+1}^n + s_i^n}{2}))}{\cosh(\rho(\frac{s_i^n - s_{i+1}^n}{2} - K))},$$

where $C(K)$ is given by (67), $\zeta_{i,\delta}(s - \frac{s_{i+1}^n + s_i^n}{2})$ generates the kernel of the linearized operator corresponding to gradient trajectory c_i as in Proposition 4.8 and $\kappa_i^n\zeta_{i,\delta}(b_i^n,\cdot) = \overline{\kappa}_i^n X_H$. In particular, we have $\kappa_i^n \to 0$, $n \to \infty$. We get

$$\int_{s_{i+1}^n + R_n}^{s_i^n - R_n} |\zeta_n(s,\theta) - \kappa_i^n\zeta_{i,\delta}|_1^p \; g_{\delta_n}(s)dsd\theta \le C_2 C(K)^p e^{(d-\rho p)(R_n - K)}.$$

The last term goes to zero as $n \to \infty$. Equality (65) follows now as in case (i). Case (iii) is entirely similar to case (ii).

In order to complete the proof of $\|\zeta_n\|_{1,\delta_n} \to 0$, $n \to \infty$, it is enough to show that $\|\zeta_n|_{I_n\times S^1}\|_{1,\delta_n} \to 0$ if $I_n =]-\infty, b_m^n - K/\delta_n]$ or $I_n = [a_0^n + K/\delta_n, +\infty[$, for any $K > 1$. The two cases are entirely similar and we give the argument only for $I_n =]-\infty, b_m^n - K/\delta_n]$. By Proposition A.2, for n sufficiently large we have $v_n(s,\theta) = \exp_{u_{\delta_n,\gamma_m,-\infty,1}(s,\theta)}(\eta_n(s,\theta))$, with $\eta_n = (\eta_n^0, \overline{\eta}_n)$, $\eta_n^0 \in W^{1,p}(I_n \times S^1, u^*_{\delta_n,\gamma_m,-\infty,1}T\widehat{W}; e^{r|s|}ds\ d\theta)$, $\overline{\eta}_n \in \overline{V}'$. Since $v_n(b_n^m,\cdot) \to \overline{\mathrm{ev}}(u_m)$ we have $\|\overline{\eta}_n\|_\infty \to 0$. Since $G_{\delta_n,\overline{\epsilon}^n}(\mathbf{v}) = u_{\delta_n,\gamma_m,-\infty,1}$ on I_n, we obtain $\zeta_n = \eta_n$, so that $\|\overline{\zeta}_n\| \to 0$. The fact that $\|\zeta_n^0\|_{1,\delta_n} \to 0$ follows from the fact that $d < r$. □

We explain now how to construct a right inverse for $D_{G_{\delta,\overline{\epsilon}}(\widetilde{w})}$ which is uniformly bounded with respect to $\delta \to 0$. The space $\widetilde{\mathcal{B}}_\delta$ is a Banach manifold whose tangent space at $\widetilde{w}$ is

$$T_{\widetilde{w}}\widetilde{\mathcal{B}}_\delta = T_{v_m}\mathcal{B}'_\delta \;{}_{d\underline{\mathrm{ev}}}\!\oplus_{d\overline{\mathrm{ev}}} T_{u_m}\mathcal{B} \;{}_{d\underline{\mathrm{ev}}}\!\oplus_{d\overline{\mathrm{ev}}} T_{v_{m-1}}\mathcal{B}'_\delta \;{}_{d\underline{\mathrm{ev}}}\!\oplus \ldots \oplus_{d\overline{\mathrm{ev}}} T_{v_0}\mathcal{B}'_\delta. \quad (68)$$

Recall that the fibered sum of two vector spaces W_1, W_2 with respect to linear maps $f_i : W_i \to W$ is the vector space

$$W_1 \;{}_{f_1}\!\oplus_{f_2} W_2 := \{(w_1, w_2) \in W_1 \oplus W_2 : f_1(w_1) = f_2(w_2)\}.$$

If $(W_1, \|\cdot\|_1)$, $(W_2, \|\cdot\|_2)$ and W are normed vector spaces, and f_1, f_2 are continuous linear maps, then $W_1 \ {}_{f_1}\!\oplus_{f_2} W_2$ is a closed subspace of $W_1 \oplus W_2$ and inherits the norm $\|\cdot\|_1 + \|\cdot\|_2$ from $W_1 \oplus W_2$. In our case

$$d\underline{\mathrm{ev}} : T_{v_m}\mathcal{B}'_\delta = W^{1,p,d} \oplus \overline{V}' \oplus \underline{V}' \to T_{\underline{\mathrm{ev}}(v_m)}S_{\gamma_m}$$

factors through the projection on $\underline{V}'$, and similarly for the other evaluation maps. Therefore the above fibered sum only affects the summands $\overline{V}$, $\underline{V}$, $\overline{V}'$, $\underline{V}'$, so that $T_{\widetilde{w}}\widetilde{\mathcal{B}}_\delta$ is a subspace of codimension $2m$ in

$$T_{v_m}\mathcal{B}'_\delta \oplus T_{u_m}\mathcal{B} \oplus T_{v_{m-1}}\mathcal{B}'_\delta \oplus \ldots \oplus T_{v_0}\mathcal{B}'_\delta.$$

As above, the norm on $T_{\widetilde{w}}\widetilde{\mathcal{B}}_\delta$ is induced from the ambient space. Recall that the $W^{1,p}$-component has weight $e^{d|s|}$ for each $T_{u_j}\mathcal{B}$, weight $e^{d||s|-s_{i,\delta}|}$ for each $T_{v_i}\mathcal{B}'_\delta$, $i = 1, \ldots, m-1$, weight $e^{d|s+s_{0,\delta}|}$ for $i = 0$ and weight $e^{d|s-s_{m,\delta}|}$ for $i = m$, with $s_{i,\delta}$ as in the definition of $g_{\delta,\overline{\epsilon}}$.

The sections $\bar{\partial}_{H,J} : \mathcal{B} \to \mathcal{E}$ and $\bar{\partial} : \mathcal{B}'_\delta \to \mathcal{E}$ defined by (41) and (56) give rise to a section over $\widetilde{\mathcal{B}}_\delta$. We denote its vertical differential by

$$D_{\widetilde{w}} : T_{\widetilde{w}}\widetilde{\mathcal{B}}_\delta \to L^{p,d}(v_m^* T\widehat{W}) \oplus L^{p,d}(u_m^* T\widehat{W}) \oplus \ldots \oplus L^{p,d}(v_0^* T\widehat{W}),$$

where

$$L^{p,d}(v_i^* T\widehat{W}) := L^p(\mathbb{R} \times S^1, v_i^* T\widehat{W}; g_{\delta,\epsilon_i,v_i}(s) ds d\theta),$$
$$L^{p,d}(u_i^* T\widehat{W}) := L^p(\mathbb{R} \times S^1, u_i^* T\widehat{W}; g_{\delta,u_i}(s) ds d\theta).$$

Lemma 4.17. *Let $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and $\{f_\gamma\} \in \mathcal{F}_{\mathrm{reg}}(H, J)$. Let $\overline{\epsilon} = (\epsilon_1, \ldots, \epsilon_{m-1})$ and let $\widetilde{w} \in \widetilde{\mathcal{B}}_\delta$ be as in Proposition 4.15. The image of the operator $D_{\widetilde{w}}$ has codimension $m-1$ and admits a complement spanned by sections $\eta_i \in L^{p,d}(v_i^* T\widehat{W})$, $i = 1, \ldots, m-1$ which are respectively supported in*

$$[-(T_i + \epsilon_i)/2\delta, -(T_i + \epsilon_i)/2\delta + 1] \times S^1 \cup [(T_i + \epsilon_i)/2\delta - 1, (T_i + \epsilon_i)/2\delta] \times S^1.$$

The operator $D_{\widetilde{w}}$ admits a right inverse $Q_{\widetilde{w}}$ defined on its image and whose norm is uniformly bounded with respect to $\delta \to 0$.

Proof. We show that

$$\operatorname{im} D_{\widetilde{w}} = \operatorname{im} D'_{v_m} \oplus \operatorname{im} D_{u_m} \oplus \operatorname{im} D'_{v_{m-1}} \oplus \ldots \oplus \operatorname{im} D'_{v_0} =: E. \tag{69}$$

By definition we have $\operatorname{im} D_{\widetilde{w}} \subset E$. Let us now choose $(x_m, y_m, \ldots, x_0) \in E$ and $\widetilde{x}_i$ and $\widetilde{y}_j$ such that $D'_{v_i}(\widetilde{x}_i) = x_i$, $D_{u_j}(\widetilde{y}_j) = y_j$. We need to modify $\widetilde{x}_i$ and $\widetilde{y}_j$ by elements lying in the kernels of the corresponding operators so that

$$d\overline{\mathrm{ev}}(\widetilde{y}_j) = d\underline{\mathrm{ev}}(\widetilde{x}_j), \quad d\underline{\mathrm{ev}}(\widetilde{y}_j) = d\overline{\mathrm{ev}}(\widetilde{x}_{j-1}), \qquad j = 1, \ldots, m. \tag{70}$$

Let us first assume $m > 1$. We have

$$T_{v_m}\mathcal{M}'_{\delta,1,-\infty}(S_{\overline{\gamma}}, S_{\overline{\gamma}}; H, J) \times T_{u_m}\mathcal{M}^{A_m}(S_{\overline{\gamma}}, S_{\gamma_{m-1}}; H, J) = \ker D'_{v_m} \times \ker D_{u_m}$$

and, because $\{f_\gamma\} \in \mathcal{F}_{\rm reg}(H,J)$, the map

$$(d\underline{\rm ev}, d\overline{\rm ev}) : \ker D'_{v_m} \times \ker D_{u_m} \to T_{\underline{\rm ev}(v_m)} S_{\overline{\gamma}} \times T_{\overline{\rm ev}(u_m)} S_{\overline{\gamma}}$$

is transverse to the diagonal. We can therefore modify $\widetilde{x}_m$ and $\widetilde{y}_m$ so that $d\overline{\rm ev}(\widetilde{y}_m) = d\underline{\rm ev}(\widetilde{x}_m)$. Similarly the map

$$(d\underline{\rm ev}, d\overline{\rm ev}) : \ker D_{u_1} \times \ker D'_{v_0} \to T_{\underline{\rm ev}(u_1)} S_{\underline{\gamma}} \times T_{\overline{\rm ev}(v_0)} S_{\underline{\gamma}}$$

is transverse to the diagonal and we can modify $\widetilde{y}_1$, $\widetilde{x}_0$ in order to achieve $d\underline{\rm ev}(\widetilde{y}_1) = d\overline{\rm ev}(\widetilde{x}_1)$. For $i = 1, \ldots, m-1$ the maps

$$(d\overline{\rm ev}, d\underline{\rm ev}) : \ker D'_{v_i} \to T_{\overline{\rm ev}(v_i)} S_{\gamma_i} \times T_{\underline{\rm ev}(v_i)} S_{\gamma_i}$$

are surjective and we can modify $\widetilde{x}_i$ so that (70) is satisfied.

If $m = 1$ the regularity hypothesis on f_γ ensures that the map

$$\begin{aligned}(d\underline{\rm ev}, d\overline{\rm ev}, d\underline{\rm ev}, d\overline{\rm ev}) \quad : \quad & \ker D'_{v_1} \times \ker D_{u_1} \times \ker D'_{v_0} \\ & \to T_{\underline{\rm ev}(v_1)} S_{\overline{\gamma}} \times T_{\overline{\rm ev}(u_1)} S_{\overline{\gamma}} \times T_{\underline{\rm ev}(u_1)} S_{\underline{\gamma}} \times T_{\overline{\rm ev}(v_0)} S_{\underline{\gamma}}\end{aligned}$$

is transverse to the product of the diagonals in the first two and in the last two factors. We can therefore modify simultaneously $\widetilde{x}_1$, $\widetilde{y}_1$, $\widetilde{x}_0$ in order to achieve (70). Therefore (69) is proved. It then follows from Proposition 4.12 that the image of $D_{\widetilde{w}}$ has codimension $m-1$ and is spanned by sections $\eta_i \in L^{p,d}(v_i^* T\widehat{W})$ supported in the desired intervals.

We now prove that $D_{\widetilde{w}}$ admits a uniformly bounded right inverse defined on its image. We observe that $D_{\widetilde{w}}$ is the restriction to ${\rm dom}(D_{\widetilde{w}})$ of the direct sum of operators $D := D'_{v_m} \oplus D_{u_m} \oplus D'_{v_{m-1}} \oplus \cdots \oplus D_{u_1} \oplus D'_{v_0}$. Let ζ_m, ζ_0 be generators of $\ker D'_{v_m}$, $\ker D'_{v_0}$ and, for $i = 1, \ldots, m-1$, let ζ_i^1, ζ_i^2 be the basis of $\ker D'_{v_i}$ constructed in Proposition 4.12. We denote by K the vector space spanned by these $2m$ sections, viewed as elements of ${\rm dom}(D)$. Then $\dim K = 2m$ and K is a complement of ${\rm dom}(D_{\widetilde{w}})$. Let $P : {\rm dom}(D) \to {\rm dom}(D_{\widetilde{w}})$ be the projection parallel to K, let Q_{u_j}, $j = 1, \ldots, m$ be uniformly bounded right inverses for D_{u_j}, let Q_{v_i}, $i = 0, \ldots, m$ be uniformly bounded right inverses for D'_{v_i} defined on their images as in Proposition 4.12, and denote $Q := Q_{v_m} \oplus Q_{u_m} \oplus Q_{v_{m-1}} \oplus \cdots \oplus Q_{u_1} \oplus Q_{u_0}$. Since $K \subset \ker D$ the operator $P \circ Q : {\rm im}(D) = {\rm im}(D_{\widetilde{w}}) \to {\rm dom}(D_{\widetilde{w}})$ is a right inverse for $D_{\widetilde{w}}$ defined on its image, and we claim that its norm is uniformly bounded for $\delta \to 0$. The norm of Q is uniformly bounded for $\delta \to 0$, so that it is enough to prove that the norm of P is uniformly bounded for $\delta \to 0$.

The sections ζ_0, ζ_m and ζ_i^1, ζ_i^2 for $i = 1, \ldots, m-1$ have the property that their respective asymptotic values (obtained by applying $d\overline{\rm ev}$ and $d\underline{\rm ev}$) are not simultaneously zero. Moreover, the same is true for any linear combination of ζ_i^1 and ζ_i^2 for $i = 1, \ldots, m-1$. As a consequence, there exists a uniform constant $C > 0$ such that, for any $\mathbf{x} = (x_m, 0, x_{m-1}, \ldots, 0, x_0) \in K$, we have

$$\|\mathbf{x}\|_{1,\delta} \le C\big(|d\underline{\rm ev}(x_m)| + |d\overline{\rm ev}(x_0)| + \sum_{i=1}^{m-1} |d\underline{\rm ev}(x_i)| + |d\overline{\rm ev}(x_i)|\big). \qquad (71)$$

Given $v \in \mathrm{dom}(D)$ we have $P(v) = v + w$ for some vector $w \in K$ which is uniquely determined by the asymptotic values of the components of v, and it follows from (71) that
$$\|w\|_{1,\delta} \le C\|v\|_{1,\delta}.$$
We obtain
$$\frac{\|P(v)\|_{1,\delta}}{\|v\|_{1,\delta}} = \frac{\|v+w\|_{1,\delta}}{\|v\|_{1,\delta}} \le 1 + C,$$
so that the norm of P is uniformly bounded by $1 + C$. This proves the Lemma. $\square$

Proposition 4.18. *Let $J \in \mathcal{J}_{\mathrm{reg}}(H)$ and $\{f_\gamma\} \in \mathcal{F}_{\mathrm{reg}}(H, J)$. Let $\widetilde{w} \in \widetilde{\mathcal{B}}_\delta$ and $\overline{\epsilon}(\delta) = (\epsilon_1(\delta), \ldots, \epsilon_{m-1}(\delta))$ be as in Proposition 4.15. The operator*

$$\begin{aligned} D_{G_{\delta,\overline{\epsilon}}(\widetilde{w})} \quad : \quad & W^{1,p}(\mathbb{R} \times S^1, G_{\delta,\overline{\epsilon}}(\widetilde{w})^* T\widehat{W}; g_{\delta,\overline{\epsilon}}(s) ds d\theta) \oplus \overline{V}'_{v_m} \oplus \underline{V}'_{v_0} \\ & \to \ L^p(\mathbb{R} \times S^1, G_{\delta,\overline{\epsilon}}(\widetilde{w})^* T\widehat{W}; g_{\delta,\overline{\epsilon}}(s) ds d\theta) \end{aligned}$$

is surjective and admits a right inverse $Q_\delta = Q_{\delta,\overline{\epsilon},\widetilde{w}}$ whose δ-norm is uniformly bounded with respect to $\delta \to 0$.

Proof. Our proof is modelled on the proof of the gluing theorem for holomorphic spheres by McDuff and Salamon [21, Ch. 10] . Let
$$v_m^\delta, u_m^\delta, v_{m-1}^\delta, \ldots, u_1^\delta, v_0^\delta$$
be the extensions of $\widehat{v}_m, \widehat{u}_m, \widehat{v}_{m-1}, \ldots, \widehat{u}_1, \widehat{v}_0$ to $\mathbb{R} \times S^1$ defined by the same formulas. Note that
$$\begin{aligned} u_j^\delta(s,\theta) &= u_j(s,\theta), \quad s \in [-R+1, R-1], \\ v_m^\delta(s,\theta) &= v_m(s,\theta), \quad\ \ s \notin [1/\delta - 1, 1/\delta], \\ v_0^\delta(s,\theta) &= v_0(s,\theta), \quad s \notin [-1/\delta, -1/\delta + 1] \end{aligned}$$
and $v_i^\delta(s,\theta) = v_i(s,\theta)$ for s outside $[-(T_i+\epsilon_i)/2\delta, -(T_i+\epsilon_i)/2\delta + 1] \cup [(T_i + \epsilon_i)/2\delta - 1, (T_i+\epsilon_i)/2\delta]$ and $i = 1, \ldots, m-1$. The difference between v_i^δ and v_i on the one hand, and that between u_j^δ and u_j on the other hand is exponentially small as $\delta \to 0$. This implies that the operators $D_{u_j^\delta}$, $D'_{v_0^\delta}$ and $D'_{v_m^\delta}$ are surjective for δ small enough and admit uniformly bounded right inverses, while the operators $D'_{v_i^\delta}$, $i = 1, \ldots, m-1$ have a codimension one image with a supplement spanned by a smooth section η_i supported in $[-(T_i+\epsilon_i)/2\delta, -(T_i+\epsilon_i)/2\delta+1] \times S^1 \cup [(T_i + \epsilon_i)/2\delta - 1, (T_i + \epsilon_i)/2\delta] \times S^1$, and admit uniformly bounded "right inverses" defined on their image. It follows that the vertical differential $D_{\widetilde{w}^\delta}$ satisfies the conclusions of Lemma 4.17, where $\widetilde{w}^\delta := (u_1^\delta, \ldots, u_m^\delta, v_0^\delta, \ldots, v_m^\delta)$. In particular, it admits a uniformly bounded right inverse defined on its image, which we denote by $Q_{\widetilde{w}^\delta}$ (see [21, Lemma 10.6.1] for a similar statement in the case of holomorphic spheres). This means that there exists a constant $c_0 > 0$ such that
$$\|Q_{\widetilde{w}^\delta}\mathbf{x}\|_{W^{1,p,d}} \le c_0 \|\mathbf{x}\|_{L^{p,d}}$$

for all $\mathbf{x} \in \operatorname{im} D_{\widetilde{w}^\delta}$ and $\delta > 0$.

We define an operator T_δ by the commutative diagram

$$\begin{tikzcd} T_{\widetilde{w}^\delta}\widetilde{\mathcal{B}}_\delta \arrow[d, "G"'] & & L^{p,d}(\widetilde{w}^{\delta *}T\widehat{W}) \arrow[ll, "Q_{\widetilde{w}^\delta}\circ P"'] \\ \operatorname{dom}(D_{G_{\delta,\overline{\epsilon}}(\widetilde{w})}) & & L^p(\mathbb{R}\times S^1, G_{\delta,\overline{\epsilon}}(\widetilde{w})^*T\widehat{W}; g_{\delta,\overline{\epsilon}}(s)dsd\theta) \arrow[ll, "T_\delta"] \arrow[u, "S"'] \end{tikzcd}$$

where

$$L^{p,d}(\widetilde{w}^{\delta *}T\widehat{W}) := L^{p,d}({v_m^\delta}^*T\widehat{W}) \oplus L^{p,d}({u_m^\delta}^*T\widehat{W}) \oplus \ldots \oplus L^{p,d}({v_0^\delta}^*T\widehat{W}).$$

In the rest of the proof we shall omit the subscript $\overline{\epsilon}$ from $G_{\delta,\overline{\epsilon}}$ and $g_{\delta,\overline{\epsilon}}$. An element of $L^{p,d}(\widetilde{w}^{\delta *}T\widehat{W})$ is denoted by

$$\mathbf{x} = (x_m, y_m, \ldots, x_0).$$

The *mixing map* P, the *splitting map* S and the *gluing map* G are defined below, and we shall prove that P, S, G are uniformly bounded with respect to $\delta \to 0$. We shall also prove that T_δ is an approximate right inverse for $D_{G_\delta(\widetilde{w})}$, i.e.

$$\|D_{G_\delta(\widetilde{w})}T_\delta\eta - \eta\|_\delta \le \frac{1}{2}\|\eta\|_\delta \tag{72}$$

for δ sufficiently small and $\eta \in L^p(\mathbb{R}\times S^1, G_\delta(\widetilde{w})^*T\widehat{W}; g_\delta(s)dsd\theta)$. This implies that $D_{G_\delta(\widetilde{w})}T_\delta$ is invertible (with the norm of its inverse bounded by 2), and $T_\delta(D_{G_\delta(\widetilde{w})}T_\delta)^{-1}$ is a right inverse for $D_{G_\delta(\widetilde{w})}$. Since P, S, G are uniformly bounded, the norm of $T_\delta(D_{G_\delta(\widetilde{w})}T_\delta)^{-1}$ is bounded by a constant multiple of $\|Q_{\widetilde{w}^\delta}\|$, hence is uniformly bounded and the conclusion of the Proposition follows.

For every $L > 0$ we fix a smooth function

$$\beta_L : \mathbb{R} \to [0,1]$$

which vanishes for $s \le 0$, which is constant equal to 1 for $s \ge L$ and whose derivative is bounded by $2/L$. We moreover require that, for L large enough, the function β_L vanishes for $s \le 1$.

We define the mixing map P. Let

$$p_i : L^{p,d}(\widetilde{w}^{\delta *}T\widehat{W}) \to \operatorname{im} D'_{v_i^\delta}, \qquad i = 0, \ldots, m$$

be the projection on $L^{p,d}((v_i^\delta)^*T\widehat{W})$ followed by the projection on $\operatorname{im} D'_{v_i^\delta}$ parallel to η_i. Recall the definition (59) of ℓ_i for $i = 0, \ldots, m$ and let

$$q_j : L^{p,d}(\widetilde{w}^{\delta *}T\widehat{W}) \to \operatorname{im} D_{u_j^\delta},$$

$$\begin{aligned} q_j(\mathbf{x})(s,\theta) \quad := \quad & y_j(s,\theta) \\ & + \beta_1(s-\ell_j)\cdot\Big((\mathbb{1}-p_j)(x_j)\Big)(s-\ell_j,\theta) \\ & + (1-\beta_1(s-\ell_{j-1}))\cdot\Big((\mathbb{1}-p_{j-1})(x_{j-1})\Big)(s-\ell_{j-1},\theta) \end{aligned}$$

for $j = 1, \dots, m$. We define

$$P : L^{p,d}(\widetilde{w}^{\delta *} T\widehat{W}) \to \operatorname{im} D_{\widetilde{w}^\delta}$$

by

$$P := p_m + q_m + p_{m-1} + \ldots + q_1 + p_0.$$

The norm of P is uniformly bounded with respect to $\delta \to 0$ since the norm of each p_i is uniformly bounded by 1.

We define now the splitting map

$$S(\eta) := \mathbf{x} = (x_m, y_m, \dots, x_0).$$

We recall the definition (58) of the catenation shifts

$$0 = s_{v_m} < s_{u_m} < s_{v_{m-1}} < \ldots < s_{u_1} < s_{v_0},$$

and set

$$x_m(s,\theta) := \beta_1(1/\delta - s)\eta(s,\theta),$$

$$x_0(s,\theta) := \beta_1(1/\delta + s)\eta(s + s_{v_0}, \theta),$$

and, for $i = 1, \dots, m-1$, $j = 1, \dots, m$,

$$y_j(s,\theta) := \begin{cases} (1 - \beta_1(-R - s))\eta(s + s_{u_j}, \theta), & s \le 0, \\ (1 - \beta_1(-R + s))\eta(s + s_{u_j}, \theta), & s \ge 0, \end{cases}$$

$$x_i(s,\theta) := \begin{cases} \beta_1((T_i + \epsilon_i)/2\delta + s)\eta(s + s_{v_i}, \theta), & s \le 0, \\ \beta_1((T_i + \epsilon_i)/2\delta - s)\eta(s + s_{v_i}, \theta), & s \ge 0. \end{cases}$$

It follows from the definition that the norm of S is uniformly bounded by 1.

We define now the gluing map $\zeta := G(\widetilde{\mathbf{x}})$, $\widetilde{\mathbf{x}} = (\widetilde{x}_m, \widetilde{y}_m, \widetilde{x}_{m-1}, \dots, \widetilde{x}_0) \in T_{\widetilde{w}^\delta}\widetilde{\mathcal{B}}_\delta$ by "slowly interpolating" the components of $\widetilde{\mathbf{x}}$. For $j = 1, \dots, m$, $i = 1, \dots, m-1$ we put

$$\zeta(s,\theta) := \begin{cases} \widetilde{x}_m(s,\theta), & -\infty < s \le 1/\delta - R/2, \\ \widetilde{y}_j(s - s_{u_j}, \theta), & s_{u_j} - R/2 \le s \le s_{u_j} + R/2, \\ \widetilde{x}_i(s - s_{v_i}, \theta), & s_{v_i} - \ell_i + 3R/2 \le s \le s_{v_i} + \ell_i - 3R/2, \\ \widetilde{x}_0(s - s_{v_0}, \theta), & s_{v_0} - 1/\delta + R/2 \le s < +\infty. \end{cases} \tag{73}$$

The above formula leaves out two types of intervals, on which the actual interpolation takes place (see Figure 6).

- If $s_{v_j} + \ell_j - 3R/2 \le s \le s_{u_j} - R/2$ (interval of length R), we define

$$\begin{aligned} \zeta(s,\theta) &:= \widetilde{x}_j(+\infty, \theta) \\ &+ (1 - \beta_{\frac{R}{2}}(s - s_{v_j} - \ell_j + R))\big(\widetilde{x}_j(s - s_{v_j}, \theta) - \widetilde{x}_j(+\infty, \theta)\big) \\ &+ (1 - \beta_{\frac{R}{2}}(-s + s_{u_j} - R))\big(\widetilde{y}_j(s - s_{u_j}, \theta) - \widetilde{y}_j(-\infty, \theta)\big). \end{aligned}$$

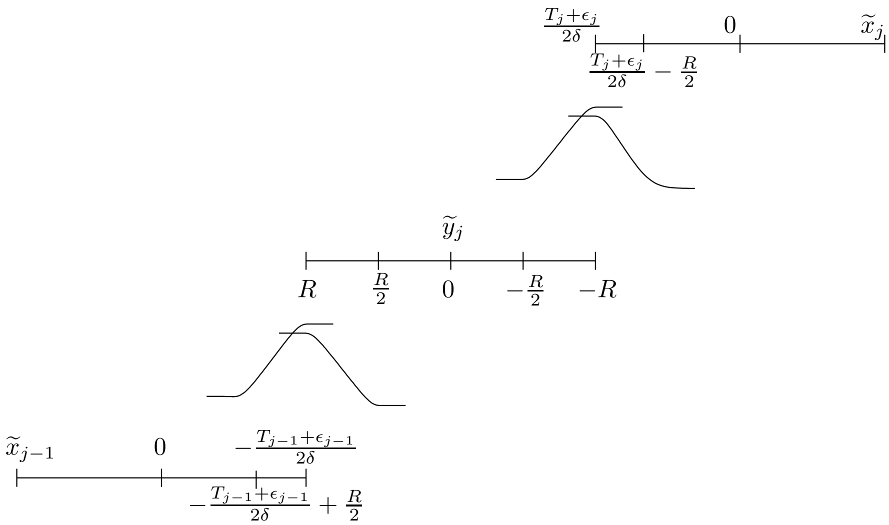

Figure 6: The gluing map G.

- If $s_{u_j} + R/2 \le s \le s_{v_{j-1}} - \ell_{j-1} + 3R/2$ (interval of length R), we define

$$\begin{aligned}\zeta(s,\theta) &:= \widetilde{x}_{j-1}(-\infty,\theta) \\ &\quad + (1-\beta_{\frac{R}{2}}(-s+s_{v_{j-1}}-\ell_{j-1}+R))\big(\widetilde{x}_{j-1}(s-s_{v_{j-1}},\theta) - \widetilde{x}_{j-1}(-\infty,\theta)\big) \\ &\quad + (1-\beta_{\frac{R}{2}}(s-s_{u_j}-R))\big(\widetilde{y}_j(s-s_{u_j},\theta) - \widetilde{y}_j(+\infty,\theta)\big).\end{aligned}$$

The section ζ is indeed of class $W^{1,p}$ because

$$\widetilde{y}_j(-\infty,\theta) = \widetilde{x}_j(+\infty,\theta), \qquad \widetilde{y}_j(+\infty,\theta) = \widetilde{x}_{j-1}(-\infty,\theta).$$

That the norm of G is uniformly bounded with respect to $\delta \to 0$ follows directly from the definition (68) of the norm on $T_{\widetilde{w}^\delta}\widetilde{\mathcal{B}}_\delta$, as well as from the definition (61) of the norm $\|\cdot\|_{1,\delta}$ on $\mathrm{dom}(D_{G_{\delta,\overline{\epsilon}}(\widetilde{w})})$ (see also Remark 4.14).

Let us now prove the estimate (72). On each of the intervals appearing in (73) we have $(D_{G_\delta(\widetilde{w})}T_\delta\eta)(s,\theta) = \eta(s,\theta)$ and we are therefore left to examine intervals of the type $[s_{v_j} + \ell_j - 3R/2, s_{u_j} - R/2]$ and $[s_{u_j} + R/2, s_{v_{j-1}} - \ell_{j-1} + 3R/2]$. We treat only the first case since the second one is entirely similar.

Upon applying the operator $D_{G_\delta(\widetilde{w})}$ to ζ we obtain five types of terms as following.

- $D_{G_\delta(\widetilde{w})}\widetilde{x}_j(+\infty,\theta)$. Since $\widetilde{x}_j(+\infty,\theta)$ does not depend on s we can view $D_{G_\delta(\widetilde{w})}$ as a family of operators on S^1. Then we have

$$\begin{aligned}\|D_{G_\delta(\widetilde{w})}\widetilde{x}_j(+\infty,\theta)\|_\delta &= \|(D_{G_\delta(\widetilde{w})} - D_{v_j(+\infty,\theta)})\widetilde{x}_j(+\infty,\theta)\|_\delta \\ &\le \|D_{G_\delta(\widetilde{w})} - D_{v_j(+\infty,\theta)}\|_\delta \|\widetilde{x}_j(+\infty,\theta)\| \\ &\le C(\delta)\|\eta\|_\delta.\end{aligned}$$

Here $D_{v_j(+\infty,\theta)}$ denotes the linearized operator at the constant cylinder $v_j(+\infty,\theta)$, the norm $\|\widetilde{x}_j(+\infty,\theta)\|$ is induced from the (1-dimensional) space $\underline{V}'_{v_j}$, and

$$C(\delta)\to 0, \qquad \delta\to 0.$$

This last statement and the last inequality follow from

$$\begin{aligned}&\|D_{G_\delta(\widetilde{w})} - D_{v_j(+\infty,\theta)}\|_\delta \\ &\le\ C(\|\widehat{v}_j - v_j(+\infty,\theta)\|_{L^{1,p,d}([(T_j+\epsilon_j)/2\delta-R/2,(T_j+\epsilon_j)/2\delta]\times S^1)} \\ &\qquad + \|\widehat{u}_j - u_j(-\infty,\theta)\|_{L^{1,p,d}([-R,-R/2]\times S^1)})\end{aligned}$$

and the fact that the intervals of integration migrate to $\pm\infty$. The above inequality makes crucial use of the fact that the weight g_δ on the necks is given by the exponential weight of the ambient spaces $\mathcal{B}_\delta$, $\mathcal{B}'_\delta$. Moreover, we have $\|\widetilde{x}_j(+\infty,\theta)\|\le\|\widetilde{\mathbf{x}}\|\le C\|\eta\|_\delta$ because $Q_{\widetilde{w}}$, P and S are uniformly bounded with respect to δ.

- $-\beta'_{R/2}(s-s_{v_j}-\ell_j+R)\big(\widetilde{x}_j(s-s_{v_j},\theta)-\widetilde{x}_j(+\infty,\theta)\big)$, as well as $\beta'_{R/2}(-s+s_{u_j}-R))\big(\widetilde{y}_j(s-s_{u_j},\theta)-\widetilde{y}_j(-\infty,\theta)\big)$. The δ-norm of each of these two terms is bounded by $C(\delta)\|\eta\|_\delta$, with $C(\delta)\to 0$ as $\delta\to 0$. To see this we first use that $|\beta'_{R/2}|\le 4/R\to 0$, $\delta\to 0$. Secondly we use that $\|\widetilde{x}_j(s-s_{v_j},\theta)-\widetilde{x}_j(+\infty,\theta)\|\le\|\widetilde{\mathbf{x}}\|\le C\|\eta\|_\delta$ and $\|\widetilde{y}_j(s-s_{u_j},\theta)-\widetilde{y}_j(-\infty,\theta)\|\le\|\widetilde{\mathbf{x}}\|\le C\|\eta\|_\delta$.
- $(1-\beta_{R/2}(s-s_{v_j}-\ell_j+R))D_{G_\delta(\widetilde{w})}\big(\widetilde{x}_j(s-s_{v_j},\theta)-\widetilde{x}_j(+\infty,\theta)\big)$ and $(1-\beta_{R/2}(-s+s_{u_j}-R))D_{G_\delta(\widetilde{w})}\big(\widetilde{y}_j(s-s_{u_j},\theta)-\widetilde{y}_j(-\infty,\theta)\big)$. The parts involving $\widetilde{x}_j(+\infty,\theta)=\widetilde{y}_j(-\infty,\theta)$ are bounded by $C(\delta)\|\eta\|_\delta$ as above. On the other hand we write

  $$D_{G_\delta(\widetilde{w})}\widetilde{x}_j = (D_{G_\delta(\widetilde{w})}-D_{\widetilde{w}^\delta})\widetilde{x}_j + D_{\widetilde{w}^\delta}\widetilde{x}_j$$

  and similarly for $D_{G_\delta(\widetilde{w})}\widetilde{y}_j$. The first term of such a sum is bounded by $C(\delta)\|\eta\|_\delta$ as above, with $C(\delta)\to 0$, $\delta\to 0$. We are left with

  $$\begin{aligned}&(1-\beta_{R/2})D_{\widetilde{w}^\delta}\widetilde{x}_j(s-s_{v_j},\theta)+(1-\beta_{R/2})D_{\widetilde{w}^\delta}\widetilde{y}_j(s-s_{u_j},\theta)\\ &\quad = \Big((P\circ S)_{v_j}\eta\Big)(s-s_{v_j},\theta)+\Big((P\circ S)_{u_j}\eta\Big)(s-s_{u_j},\theta) \quad = \quad \eta.\end{aligned}$$

  Here we denote by $(P\circ S)_{v_j}$, $(P\circ S)_{u_j}$ the components of $P\circ S$ in $L^{p,d}(v_j^{\delta *}T\widehat{W})$ and $L^{p,d}(u_j^{\delta *}T\widehat{W})$ respectively. The first equality uses the fact that $1-\beta_{R/2}\equiv 1$ on the support of $(P\circ S)_{v_j}\eta$ and on the support of $(P\circ S)_{u_j}\eta$, as well as $D_{\widetilde{w}^\delta}\circ Q_{\widetilde{w}^\delta}=\mathbb{1}$.

As a conclusion we have

$$\|D_{G_\delta(\widetilde{w})}T_\delta\eta-\eta\|_\delta\le C(\delta)\|\eta\|_\delta, \qquad C(\delta)\to 0,\ \delta\to 0,$$

and the estimate (72) holds for δ small enough. $\square$

We shall use the following quantitative form of the implicit function theorem from McDuff and Salamon [21, A.3.4].

Theorem 4.19. *Let X and Y be Banach spaces, $U \subset X$ be an open set, and $f : U \to Y$ be a continuously differentiable map. Let $x_0 \in U$ be such that $D := df(x_0) : X \to Y$ is surjective and has a bounded right inverse $Q : Y \to X$. Choose positive constants ε and c such that $\|Q\| \le c$, $B_\varepsilon(x_0) \subset U$, and*

$$\|x - x_0\| < \varepsilon \quad \Longrightarrow \quad \|df(x) - D\| \le 1/2c. \tag{74}$$

Then, for any $x_1 \in X$ satisfying

$$\|f(x_1)\| < \varepsilon/4c, \qquad \|x_1 - x_0\| < \varepsilon/8, \tag{75}$$

there exists a unique $x \in X$ such that

$$f(x) = 0, \quad x - x_1 \in \operatorname{im} Q, \quad \|x - x_0\| \le \varepsilon. \tag{76}$$

Moreover, $\|x - x_1\| \le 2c\|f(x_1)\|$.

The above theorem will be used within the following setup. Consider an element $[\mathbf{u}] \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ and denote $u_0 := G_{\delta,\overline{\epsilon}}(\mathbf{u})$. Given $\varepsilon > 0$ we denote by $B_\varepsilon(0)$ the ball of radius ε centered at 0 in $W^{1,p}(\mathbb{R}\times S^1, u_0^*T\widehat{W}; \|\cdot\|_{1,\delta})$, where $\|\cdot\|_{1,\delta}$ is defined by (61). For $\zeta \in W^{1,p}(\mathbb{R}\times S^1, u_0^*T\widehat{W}; \|\cdot\|_{1,\delta})$ we write

$$\begin{aligned}
\zeta \quad = \quad & \zeta_1 + \sum_{j=1}^{m} \overline{\kappa}_j \beta(-s + s_{u_j})\beta(s - s_{u_j} + 2R) X_H \\
& + \sum_{j=1}^{m} \underline{\kappa}_j \beta(s - s_{u_j})\beta(-s + s_{u_j} + 2R) X_H \\
& + \sum_{i=1}^{m-1} \kappa_i \beta(s - s_{v_i} + \ell_i - 2R)\beta(-s + s_{v_i} + \ell_i - 2R)\zeta_{i,\delta}(\cdot - s_{v_i}, \cdot)
\end{aligned}$$

with $\ell_i = R + (T_i + \epsilon_i)/2\delta$ and $\zeta_{i,\delta}$ the generator of ker D_{v_i} whose value at 0 is the vector field X_H along γ_i. Then

$$\|\zeta\|_{1,\delta} = \|\zeta_1\|_{W^{1,p}(g_{\delta,\overline{\epsilon}})} + \sum_{j=1}^{m}(|\overline{\kappa}_j| + |\underline{\kappa}_j|) + \sum_{i=1}^{m-1} |\kappa_i|.$$

We denote

$$\widetilde{\zeta} := \zeta_1 + \sum_{j=1}^{m} \big(\overline{\kappa}_j \beta(-s+s_{u_j})\beta(s-s_{u_j}+2R) X_H + \underline{\kappa}_j \beta(s-s_{u_j})\beta(-s+s_{u_j}+2R) X_H\big),$$

so that $\widetilde{\zeta}(s_{v_i}, \cdot)$ is L^2-orthogonal to $\zeta_{i,\delta}(0, \cdot)$. For each $i = 1, \ldots, m-1$ we consider the smooth cutoff function

$$\rho_{i,\delta,\overline{\epsilon}}(s) := \beta(s - s_{v_i} + \ell_i - 2R)\beta(-s + s_{v_i} + \ell_i - 2R),$$

so that $\rho_{i,\delta,\overline{\epsilon}}$ vanishes outside $[s_{v_i} - \frac{T_i+\epsilon_i}{2\delta}, s_{v_i} + \frac{T_i+\epsilon_i}{2\delta}]$ and $\rho_{i,\delta,\overline{\epsilon}} \equiv 1$ on the interval $[s_{v_i} - \frac{T_i+\epsilon_i}{2\delta} + 1, s_{v_i} + \frac{T_i+\epsilon_i}{2\delta} - 1]$.

We define $\varphi_\zeta(u_0) : \mathbb{R} \times S^1 \to \widehat{W}$ by

$$\varphi_\zeta(u_0)(s,\theta) := \begin{cases} u_0(s,\theta), & s_{u_j} - R \le s \le s_{u_j} + R, \\ \varphi^{f_{\gamma_i}}_{\rho_{i,\delta,\overline{\epsilon}}(s)\kappa_i}(u_0(s,\cdot))(\theta), & s_{v_i} - \frac{T_i+\epsilon_i}{2\delta} \le s \le s_{v_i} + \frac{T_i+\epsilon_i}{2\delta}. \end{cases}$$

Note that the last formula can also be written in the chart (ϑ, z) around S_{γ_i} as $\vartheta \circ \varphi_\zeta(u_0)(s,\theta) = \vartheta \circ \varphi^{f_{\gamma_i}}_{\rho_{i,\delta,\overline{\epsilon}}(s)\kappa_i}(u_0(s,0)) + \theta$. Given a vector field ξ along u_0 we define the vector field $\varphi_{\zeta *}\xi$ along $\varphi_\zeta(u_0)$ by

$$\varphi_{\zeta *}\xi(s,\theta) := \begin{cases} \xi(s,\theta), & s_{u_j} - R \le s \le s_{u_j} + R, \\ \varphi^{f_{\gamma_i}}_{\rho_{i,\delta,\overline{\epsilon}}(s)\kappa_i *}(u_0)\xi(s,\theta), & s_{v_i} - \frac{T_i+\epsilon_i}{2\delta} \le s \le s_{v_i} + \frac{T_i+\epsilon_i}{2\delta}. \end{cases}$$

We define a map

$$\Phi : B_\varepsilon(0) \to \mathcal{B}_\delta = \mathcal{B}^{1,p,d}_\delta(\overline{\gamma}_p, \underline{\gamma}_q, A; H, \{f_\gamma\}) \tag{77}$$

by

$$\Phi(\zeta) := \exp_{\varphi_\zeta(u_0)}(\varphi_{\zeta *}\widetilde{\zeta}).$$

Since $\rho_{i,\delta,\overline{\epsilon}}$ is precisely the coefficient of $\zeta_{i,\delta}$ in our splitting for ζ, it follows that $d\Phi(0) = \mathrm{Id}$. Hence, for $\varepsilon > 0$ small enough the map Φ is a diffeomorphism onto its image, i.e. a chart.

We denote $X := W^{1,p}(\mathbb{R} \times S^1, u_0^* T\widehat{W}; \|\cdot\|_{1,\delta})$, $U := B_\varepsilon(0) \subset X$, $Y := L^p(\mathbb{R} \times S^1, u_0^* T\widehat{W}; g_{\delta,\overline{\epsilon}} ds d\theta)$, $x_0 = 0$. For $\varepsilon > 0$ small enough the Banach bundle $\mathcal{E} \to \mathcal{B}_\delta$ can be trivialized over the image of Φ as $B_\varepsilon(0) \times Y$, and we denote by $f : B_\varepsilon(0) \to Y$ the section $\bar\partial_{H_\delta,J} \circ \Phi$ read in this trivialization. Then $df(0) = D_{u_0}$ is surjective and has a right inverse Q_δ whose δ-norm is uniformly bounded with respect to $\delta \to 0$ by Proposition 4.18. In order for the hypotheses of Theorem 4.19 to be satisfied we need to check that (74) holds.

Lemma 4.20. *There exists a constant $C > 0$ independent of δ such that, for all $x \in B_\varepsilon(0)$, we have*

$$\|df(x) - df(0)\| \le C\|x\|_{1,\delta}.$$

Remark 4.21. The motivation for introducing the chart Φ is that we must use the "compensated" norm $\|\cdot\|_{1,\delta}$. The lemma would fail if one used the usual exponential chart $\zeta \mapsto \exp_{u_0}(\zeta)$ instead of Φ, because the estimate for the expression (82) in the proof below would not hold.

Proof. We need to prove the existence of a uniform constant $C > 0$ such that

$$\|D(\bar\partial_{H_\delta,J} \circ \Phi)(x) \cdot \zeta - D(\bar\partial_{H_\delta,J} \circ \Phi)(0) \cdot \zeta\|_\delta \le C\|x\|_{1,\delta}\|\zeta\|_{1,\delta} \tag{78}$$

for all $\zeta \in X$. We recall the decomposition $\zeta = \widetilde{\zeta} + \sum_{i=1}^{m-1} \kappa_i \rho_{i,\delta,\overline{\epsilon}} \zeta_{i,\delta}$, which satisfies $\|\zeta\|_{1,\delta} = \|\widetilde{\zeta}\|_{1,\delta} + \sum_{i=1}^{m-1} |\kappa_i|$. It is therefore enough to prove (78) separately for $\zeta = \widetilde{\zeta}$ and for $\zeta = \rho_{i,\delta,\overline{\epsilon}} \zeta_{i,\delta}$, $i = 1, \ldots, m-1$. We abbreviate in the following computations $\bar{\partial} = \bar{\partial}_{H_\delta, J}$.

We first assume $\zeta = \rho_{i,\delta,\overline{\epsilon}} \zeta_{i,\delta}$. Given $x = \widetilde{x} + \sum_{j=1}^{m-1} \kappa_j \rho_{j,\delta,\overline{\epsilon}} \zeta_{j,\delta}$ we have

$$\begin{aligned} & D(\bar{\partial} \circ \Phi)(x)\zeta - D(\bar{\partial} \circ \Phi)(0)\zeta \\ = \quad & D(\bar{\partial} \circ \Phi)(x)\zeta - D(\bar{\partial} \circ \Phi)(\sum_{j=1}^{m-1} \kappa_j \rho_{j,\delta,\overline{\epsilon}} \zeta_{j,\delta})\zeta \qquad (79) \\ & + D(\bar{\partial} \circ \Phi)(\sum_{j=1}^{m-1} \kappa_j \rho_{j,\delta,\overline{\epsilon}} \zeta_{j,\delta})\zeta - D(\bar{\partial} \circ \Phi)(0)\zeta. \qquad (80) \end{aligned}$$

The term (79) is further equal to

$$\begin{aligned} & \frac{d}{dt}\Big|_{t=0} \bar{\partial}(\exp_{\varphi_{x+t\zeta}(u_0)}(\varphi_{x+t\zeta *}\widetilde{x})) - \frac{d}{dt}\Big|_{t=0} \bar{\partial}(\exp_{\varphi_{x+t\zeta}(u_0)}(0)) \\ & = D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) \cdot \nabla_t \varphi_{x+t\zeta *}\widetilde{x} \\ & \quad + D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) \cdot \rho_{i,\delta,\overline{\epsilon}} \nabla f_{\gamma_i}(\varphi_x(u_0)) \\ & \quad - D_{\varphi_x(u_0)} \cdot D_1 \exp_{\varphi_x(u_0)}(0) \cdot \rho_{i,\delta,\overline{\epsilon}} \nabla f_{\gamma_i}(\varphi_x(u_0)) \\ & = D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) \cdot \nabla_t \varphi_{x+t\zeta *}\widetilde{x} \qquad (81) \\ & \quad + D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot (D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0)) \\ & \qquad \qquad \cdot \rho_{i,\delta,\overline{\epsilon}} \nabla f_{\gamma_i}(\varphi_x(u_0)) \\ & \quad + (D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)}) \cdot D_1 \exp_{\varphi_x(u_0)}(0) \cdot \rho_{i,\delta,\overline{\epsilon}} \nabla f_{\gamma_i}(\varphi_x(u_0)). \end{aligned}$$

Here $\mathcal{T}$ is the parallel transport in $\widehat{W}$ along the geodesic $\tau \mapsto \exp_{\varphi_x(u_0)}(\tau \varphi_{x*}\widetilde{x})$, $\tau \in [0,1]$, and we have $\rho_{i,\delta,\overline{\epsilon}} \nabla f_{\gamma_i}(\varphi_x(u_0)) = \rho_{i,\delta,\overline{\epsilon}} (\varphi^{f_{\gamma_i}}_{\rho_{i,\delta,\overline{\epsilon}} \kappa_i})_* \zeta_{i,\delta}$.

We study the first term in (81). We have pointwise bounds

$$|\nabla_t \varphi_{x+t\zeta *}\widetilde{x}| \le C(1 + |\kappa_i|)|\widetilde{x}|,$$
$$|\nabla \nabla_t \varphi_{x+t\zeta *}\widetilde{x}| \le C(1 + |\kappa_i|)(|\widetilde{x}| + |\nabla \widetilde{x}|)$$

for some universal constant $C > 0$. In particular

$$\|\nabla_t \varphi_{x+t\zeta *}\widetilde{x}\|_{W^{1,p}(g_{\delta,\overline{\epsilon}})} \le C \|\widetilde{x}\|_{1,\delta}$$

if $|\kappa_i| \le \|x\|_{1,\delta} \le \varepsilon$, with $C > 0$ a universal constant. On the other hand the operators $D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) : W^{1,p}(g_{\delta,\overline{\epsilon}}) \to W^{1,p}(g_{\delta,\overline{\epsilon}})$ and $D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} : W^{1,p}(g_{\delta,\overline{\epsilon}}) \to L^p(g_{\delta,\overline{\epsilon}})$ are uniformly bounded if $\|x\|_\infty \le C\|x\|_{1,\delta} \le C\varepsilon$ (we use here the Sobolev inequality). This implies that the δ-norm of the first term in (81) is bounded by a constant multiple of $\|\widetilde{x}\|_{1,\delta}$.

We now study the second term in (81). Let $|||\cdot|||$ be the operator norm for continuous linear maps

$$W^{1,p}(\varphi_x(u_0)^* T\widehat{W}; \|\cdot\|_{1,\delta}) \to W^{1,p}(\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})^* T\widehat{W}; g_{\delta,\overline{\epsilon}} ds d\theta).$$

We claim that $|||D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0)||| \le C\|\widetilde{x}\|_{1,\delta}$ for some uniform constant $C > 0$, provided $\|x\|_{1,\delta} \le \varepsilon$. Indeed, since the metric on $\widehat{W}$ varies smoothly, for any $\xi = \widetilde{\xi} + \sum_{\ell=1}^{m-1} \kappa'_\ell \rho_{\ell,\delta,\overline{\epsilon}} \nabla f_{\gamma_\ell}(\varphi_x(u_0))$ we have pointwise bounds

$$\big|(D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0))\widetilde{\xi}\big| \le C|\widetilde{x}||\widetilde{\xi}|,$$

$$\big|\nabla(D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0))\widetilde{\xi}\big| \le C(|\nabla\widetilde{x}||\widetilde{\xi}| + |\widetilde{x}||\nabla\widetilde{\xi}|),$$

$$\big|(D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0))\rho_{\ell,\delta,\overline{\epsilon}} \nabla f_{\gamma_\ell}(\varphi_x(u_0))\big| \le C|\widetilde{x}|,$$

$$\big|\nabla(D_1 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_1 \exp_{\varphi_x(u_0)}(0))\rho_{\ell,\delta,\overline{\epsilon}} \nabla f_{\gamma_\ell}(\varphi_x(u_0))\big| \le C(|\widetilde{x}| + |\nabla\widetilde{x}|).$$

The claim then follows by integration with respect to the weight $g_{\delta,\overline{\epsilon}}$ and by using the Sobolev inequalities $\|\widetilde{x}\|_{L^\infty} \le C\|\widetilde{x}\|_{1,\delta}$ and $\|\widetilde{\xi}\|_{L^\infty} \le C\|\widetilde{\xi}\|_{1,\delta}$. On the other hand, as already seen above, the operator $D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})}$ acting from the space $W^{1,p}(g_{\delta,\overline{\epsilon}})$ to $L^p(g_{\delta,\overline{\epsilon}})$ is uniformly bounded for $\|x\|_{1,\delta} \le \varepsilon$, since its coefficients are bounded. We infer that the δ-norm of the second term in (81) is bounded by a constant multiple of $\|\widetilde{x}\|_{1,\delta}$.

We finally study the third term in (81). We claim that $\|D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)}\| \le C\|\widetilde{x}\|_{1,\delta}$ for some uniform constant $C > 0$, provided $\|x\|_{1,\delta} \le \varepsilon$. This follows from the pointwise bounds

$$\big|(D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)})\widetilde{\xi}\big| \le C|\widetilde{x}|(|\widetilde{\xi}| + |\nabla\widetilde{\xi}|),$$

$$\big|(D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)})\rho_{\ell,\delta,\overline{\epsilon}} \nabla f_{\gamma_\ell}(\varphi_x(u_0))\big| \le C|\widetilde{x}|$$

by integrating with respect to the weight $g_{\delta,\overline{\epsilon}}$ and by using the previous Sobolev inequalities. Since $D_1 \exp_{\varphi_x(u_0)}(0) = \mathrm{Id}$, we infer that the δ-norm of the third term in (81) is bounded by a constant multiple of $\|\widetilde{x}\|_{1,\delta}$.

As a conclusion, the δ-norm of the expression in (79) is bounded by a constant multiple of $\|\widetilde{x}\|_{1,\delta}$.

We now consider the expression in (80), which can be written as

$$\begin{aligned} & D(\bar\partial \circ \Phi)(\kappa_i \rho_{i,\delta,\overline{\epsilon}} \zeta_{i,\delta})\zeta - D(\bar\partial \circ \Phi)(0)\zeta \\ = \ & \frac{d}{dt}\Big|_{t=0} \bar\partial(\varphi_{\kappa_i\zeta + t\zeta}(u_0)) - \frac{d}{dt}\Big|_{t=0} \bar\partial(\varphi_{t\zeta}(u_0)) \\ = \ & \frac{d}{dt}\Big|_{t=0} \bar\partial(u_0(\cdot + (\kappa_i + t)\rho_{i,\delta,\overline{\epsilon}}, \cdot)) - \frac{d}{dt}\Big|_{t=0} \bar\partial(u_0(\cdot + t\rho_{i,\delta,\overline{\epsilon}}, \cdot)). \end{aligned} \tag{82}$$

Each term in the above difference is supported in the intervals $[s_{v_i} - \frac{T_i+\epsilon_i}{2\delta}, s_{v_i} - \frac{T_i+\epsilon_i}{2\delta} + 1]$ and $[s_{v_i} + \frac{T_i+\epsilon_i}{2\delta} - 1, s_{v_i} + \frac{T_i+\epsilon_i}{2\delta}]$. Moreover, their difference is pointwise bounded by $C|\kappa_i|$ for some uniform constant $C > 0$. Since the weight $g_{\delta,\overline{\epsilon}}$ is uniformly bounded on the above intervals of length 1, we infer that the δ-norm of the expression in (80) is bounded by $C|\kappa_i|$, hence by $C\|x\|_{1,\delta}$ for some uniform constant $C > 0$.

We now assume $\zeta = \widetilde{\zeta}$ and we again decompose $D(\bar{\partial} \circ \Phi)(x)\zeta - D(\bar{\partial} \circ \Phi)(0)\zeta$ as the sum of the expressions in (79) and (80).

The expression in (79) can be written

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \bar{\partial}(\exp_{\varphi_x(u_0)}(\varphi_{x*}(\widetilde{x} + t\widetilde{\zeta}))) - \frac{d}{dt}\Big|_{t=0} \bar{\partial}(\exp_{\varphi_x(u_0)}(\varphi_{x*} t\widetilde{\zeta})) \\
&\quad = \quad D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) \cdot \varphi_{x*}\widetilde{\zeta} \\
&\qquad\qquad\qquad\qquad - D_{\varphi_x(u_0)} \cdot D_2 \exp_{\varphi_x(u_0)}(0) \cdot \varphi_{x*}\widetilde{\zeta} \\
&\quad = \quad D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot (D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_2 \exp_{\varphi_x(u_0)}(0)) \cdot \varphi_{x*}\widetilde{\zeta} \\
&\qquad\quad + (D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)}) \cdot D_2 \exp_{\varphi_x(u_0)}(0) \cdot \varphi_{x*}\widetilde{\zeta}. \qquad (83)
\end{aligned}$$

Here $\mathcal{T}$ denotes the same parallel transport map as above.

We claim that the δ-norm of the first term in the expression (83) is bounded by $C\|\widetilde{x}\|_{1,\delta}\|\widetilde{\zeta}\|_{1,\delta}$ when $\|x\|_{1,\delta} \le \varepsilon$, for some uniform constant $C > 0$. We have the pointwise estimates

$$|\varphi_{x*}\widetilde{\zeta}| \le C(1 + \sum_{j=1}^{m-1} |\kappa_j|)|\widetilde{\zeta}|,$$

$$|\nabla\varphi_{x*}\widetilde{\zeta}| \le C(1 + \sum_{j=1}^{m-1} |\kappa_j|)(|\widetilde{\zeta}| + |\nabla\widetilde{\zeta}|),$$

which imply $\|\varphi_{x*}\widetilde{\zeta}\|_{W^{1,p}(g_{\delta,\bar{\epsilon}})} \le C\|\widetilde{\zeta}\|_{1,\delta}$ for some uniform constant $C > 0$, provided $\|x\|_{1,\delta} \le \varepsilon$. On the other hand, the pointwise estimates

$$\left|(D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_2 \exp_{\varphi_x(u_0)}(0))\xi\right| \le C|\widetilde{x}||\xi|,$$

$$\left|\nabla(D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_2 \exp_{\varphi_x(u_0)}(0))\xi\right| \le C(|\nabla\widetilde{x}||\xi| + |\widetilde{x}||\nabla\xi|)$$

show that the norm of the operator $D_2 \exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x}) - \mathcal{T} \cdot D_2 \exp_{\varphi_x(u_0)}(0)$ acting from $W^{1,p}(g_{\delta,\bar{\epsilon}})$ to itself is bounded by $C\|\widetilde{x}\|_{1,\delta}$. Finally, we have already seen that the operator $D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})}$ acting between $W^{1,p}(g_{\delta,\bar{\epsilon}})$ and $L^p(g_{\delta,\bar{\epsilon}})$ is uniformly bounded, and the claim follows.

We now claim that the δ-norm of the second term in the expression (83) is also bounded by $C\|\widetilde{x}\|_{1,\delta}\|\widetilde{\zeta}\|_{1,\delta}$ when $\|x\|_{1,\delta} \le \varepsilon$, for some uniform constant $C > 0$. We have the pointwise estimate

$$\left|(D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)})\xi\right| \le C|\widetilde{x}|(|\xi| + |\nabla\xi|),$$

which implies that the norm of the operator $D_{\exp_{\varphi_x(u_0)}(\varphi_{x*}\widetilde{x})} \cdot \mathcal{T} - D_{\varphi_x(u_0)}$ acting from $W^{1,p}(g_{\delta,\bar{\epsilon}})$ to $L^p(g_{\delta,\bar{\epsilon}})$ is bounded by $C\|\widetilde{x}\|_{1,\delta}$ for some uniform constant $C > 0$. Since $\|\varphi_{x*}\widetilde{\zeta}\|_{W^{1,p}(g_{\delta,\bar{\epsilon}})} \le C\|\widetilde{\zeta}\|_{1,\delta}$ and $D_2 \exp_{\varphi_x(u_0)}(0) = \mathrm{Id}$, the claim follows.

We finally study the term (80) in the decomposition of $D(\bar{\partial}\circ\Phi)(x)\zeta - D(\bar{\partial}\circ\Phi)(0)\zeta$, which can be written

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\bar{\partial}(\exp_{\varphi_x(u_0)}\varphi_{x*}t\widetilde{\zeta}) - \frac{d}{dt}\Big|_{t=0}\bar{\partial}(\exp_{u_0} t\widetilde{\zeta}) \\
&= D_{\varphi_x(u_0)}\cdot D_2\exp_{\varphi_x(u_0)}(0)\cdot\varphi_{x*}\widetilde{\zeta} - D_{u_0}\cdot D_2\exp_{u_0}(0)\cdot\widetilde{\zeta} \\
&= D_{\varphi_x(u_0)}\cdot\varphi_{x*}\widetilde{\zeta} - D_{u_0}\cdot\widetilde{\zeta}.
\end{aligned}$$

This last expression is pointwise bounded by $C(\sum_{j=1}^{m-1}|\kappa_j|)(|\widetilde{\zeta}|+|\nabla\widetilde{\zeta}|)$, which implies that its δ-norm is bounded by $C\|x\|_{1,\delta}\|\widetilde{\zeta}\|_{1,\delta}$ for some uniform constant $C>0$.

This proves the lemma. □

Proposition 4.22. *Let $[\mathbf{u}]\in\mathcal{M}^A(p,q;H,\{f_\gamma\},J)$. There exists $\delta_1>0$ and a one-parameter family $[u_\delta]\in\mathcal{M}^A(\overline{\gamma}_p,\underline{\gamma}_q;H_\delta,J)$, $0<\delta<\delta_1$ such that*

$$[u_\delta]\to[\mathbf{u}],\quad \delta\to 0.$$

Here convergence is understood in the sense of Definition 4.2. Moreover, if $\dim\mathcal{M}^A(p,q;H,\{f_\gamma\},J)=0$ *then the intermediate gradient fragments in* $[\mathbf{u}]$ *are nonconstant and the above one-parameter family is unique.*

Remark 4.23. The fact that the intermediate gradient fragments in $[\mathbf{u}]$ are nonconstant is the reason why we had to prove the gluing theorem only in the case where the intermediate lengths of gradient trajectories are strictly positive: $T_i>0$, $i=1,\dots,m-1$, where m is the number of sublevels in $[\mathbf{u}]$.

Proof. We choose a representative $\mathbf{u}=(c_m,u_m,\dots,u_1,c_0)$ of $[\mathbf{u}]$ and we apply Theorem 4.19 in a chart of $\mathcal{B}_\delta$ as above. By Proposition 4.18 the operator D admits a right inverse Q_δ which is uniformly bounded with respect to δ by some constant c. By Lemma 4.20 there exists $\varepsilon>0$ independent of δ such that condition (74) is satisfied. We set $x_0:=G_\delta(\mathbf{u})$. By Proposition 4.15 we have

$$\lim_{\delta\to 0}\|f(x_0)\|=0$$

and therefore condition (75) is satisfied on some open neighbourhood of x_0 if δ is small enough. Taking $x_1:=x_0$ in the statement of Theorem 4.19 provides us with an element $x\in X$ satisfying (76). We set

$$u_\delta:=x.$$

Then $[u_\delta]\in\mathcal{M}^A(\overline{\gamma}_p,\underline{\gamma}_q;H_\delta,J)$. Because $\|x-x_0\|\le 2c\|f(x_0)\|\to 0$ and $x_0=G_\delta(\mathbf{u})\to\mathbf{u}$ by construction, we infer by Proposition 4.16 that $[u_\delta]\to[\mathbf{u}]$, $\delta\to 0$.

We now assume that the dimension of $\mathcal{M}^A(p,q;H,\{f_\gamma\},J)$ is zero. We have

$$\begin{aligned}
&\dim\mathcal{M}^A(\overline{\gamma}_p,\underline{\gamma}_q;H_\delta,J)= \\
&= \mu(\overline{\gamma}_p)-\mu(\underline{\gamma}_q)+2\langle c_1(TW),A\rangle-1 \\
&= \mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle-1+\mathrm{ind}(p)-\mathrm{ind}(q),
\end{aligned}$$

hence $\mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle = 1-\mathrm{ind}(p)+\mathrm{ind}(q)\le 2$. On the other hand $\mu(\overline{\gamma})-\mu(\underline{\gamma})+2\langle c_1(TW),A\rangle = \sum_{i=1}^m \mu(\gamma_i)-\mu(\gamma_{i-1})+2\langle c_1(TW),A_i\rangle$, where $m\ge 0$ is the number of sublevels of $\mathbf{u}$. Each of the summands is nonnegative by transversality, and the only possibilities occuring are the following:

(i) each summand is zero, which means that all the Floer trajectories involved in $\mathbf{u}$ are rigid;

(ii) one of the summands is 1 and the others vanish. Since $[\mathbf{u}]$ is rigid, the only nonrigid summand must be u_0 or u_m, while c_0, respectively c_m have to be constant.;

(iii) two of the summands are 1, and the others vanish. As above, the nonrigid summands must be u_0 and u_m, while c_0 and c_m are constant;

(iv) one of the summands is 2 and the others vanish. Since $[\mathbf{u}]$ is rigid we must have $m=1$ and c_0, c_1 have to be constant.

In each of the cases (i-iii) the intermediate gradient trajectories have to be nonconstant by transversality of the evaluation maps (34).

Let now $[\widetilde{v}_\delta]\to[\mathbf{u}]$, $\delta\to 0$. Since the only possible intermediate gradient trajectory in $[\mathbf{u}]$ is nonconstant, we can apply Proposition 4.16. We obtain representatives $v_\delta\in[\widetilde{v}_\delta]$, $\mathbf{v}\in[\mathbf{u}]$ and functions $\overline{\epsilon}=\overline{\epsilon}(\delta)=(\epsilon_1(\delta),\ldots,\epsilon_{m-1}(\delta))$ such that v_δ, $G_{\delta,\overline{\epsilon}}(\mathbf{v})$ belong to some $\|\cdot\|_{1,\delta,\overline{\epsilon}}$ -chart in $\mathcal{B}_\delta$ and

$$\|v_\delta - G_{\delta,\overline{\epsilon}}(\mathbf{v})\|_{1,\delta,\overline{\epsilon}}\to 0,\quad \delta\to 0. \tag{84}$$

We have to prove that u_δ and v_δ differ by a shift for $\delta>0$ sufficiently small.

Let us choose a continuous path $\mathbf{v}_t\in\widehat{\mathcal{M}}^A(p,q;H,\{f_\gamma\},J)$, $t\in[0,1]$ with $\mathbf{v}_0=\mathbf{u}$, $\mathbf{v}_1=\mathbf{v}$. We denote $y_t=y_t(\delta):=G_{\delta,t\overline{\epsilon}}(\mathbf{v}_t)$ and $\|\cdot\|_t:=\|\cdot\|_{1,\delta,t\overline{\epsilon}}$. Note that, for each $\delta>0$, there exists a continuous function $C_\delta:[0,1]\times[0,1]\to\mathbb{R}^+$ such that

$$\frac{1}{C_\delta(t,t')}\|\cdot\|_{t'}\le\|\cdot\|_t\le C_\delta(t,t')\|\cdot\|_{t'},\quad t,t'\in[0,1]$$

satisfying $C_\delta(t,t)=1$, $t\in[0,1]$.

As $y_t(\delta)$ and $D_{y_t(\delta)}$ vary continuously with t and δ, we can choose $\varepsilon>0$ and $c>0$ so that the hypotheses of the implicit function theorem 4.19 are satisfied for each $y_t(\delta)$, $t\in[0,1]$, $0<\delta<\delta_1$ and some suitable constant $\delta_1>0$. After further shrinking δ_1 we can also assume that $\|f(y_t)\|_t<\varepsilon/4c$ for all $t\in[0,1]$ and $0<\delta<\delta_1$. Finally, in view of (84), for some smaller δ_1 we can achieve $\|v_\delta-y_1(\delta)\|_1\le\varepsilon$.

We define

$$I_\delta:=\{t\in[0,1]\,:\,\exists\, x\in[v_\delta],\ \|x-y_t(\delta)\|_t\le\varepsilon\},\qquad 0<\delta<\delta_1.$$

We prove that $I_\delta=[0,1]$ by showing that it is a nonempty open and closed subset of $[0,1]$. Note that I_δ is nonempty since $1\in I_\delta$. We prove now that I_δ

is closed. Assume that $t_n \in I_\delta$ is such that $t_n \to t$. Let $x_n \in [v_\delta]$ be such that $\|x_n - y_{t_n}(\delta)\|_{t_n} \le \varepsilon$. By the triangular inequality we see that $\|x_n - y_t(\delta)\|_t$ stays bounded, hence the sequence of shifts defining x_n is also bounded and, up to a subsequence, we may assume that $x_n \to x \in [v_\delta]$. Then

$$\begin{aligned}\|x_n - y_t(\delta)\|_t &\le C_\delta(t,t_n)\|x_n - y_t(\delta)\|_{t_n} \\ &\le C_\delta(t,t_n)(\|x_n - y_{t_n}(\delta)\|_{t_n} + \|y_{t_n}(\delta) - y_t(\delta)\|_{t_n}) \\ &\le C_\delta(t,t_n)(\varepsilon + \|y_{t_n}(\delta) - y_t(\delta)\|_{t_n}).\end{aligned}$$

We pass to the limit $n \to \infty$ and obtain $\|x - y_t(\delta)\|_t \le \varepsilon$, hence $t \in I_\delta$. We prove now that I_δ is open. Let $t \in I_\delta$ and choose an open interval J containing t such that $C_\delta(t', t) < 2$ and $\|y_{t'}(\delta) - y_t(\delta)\|_t < \varepsilon/8$ for all $t' \in J$. Theorem 4.19 applied to $x_0 := y_t(\delta)$ and $x_1 := y_{t'}(\delta)$ yields x such that $f(x) = 0$ and $\|x - y_t(\delta)\|_t \le \varepsilon$. The uniqueness statement in the implicit function theorem ensures that the intersection of the space of solutions with the $\|\cdot\|_t$-ball of radius ε centered at x_0 is a graph over $\ker D$. Since $\dim\ker D = 1$ and since translation in the s-variable already provides a 1-parameter family of solutions, we infer that $x \in [v_\delta]$. Moreover, the last statement in Theorem 4.19 gives $\|x - y_{t'}(\delta)\|_t \le \varepsilon/2$. Then $\|x - y_{t'}(\delta)\|_{t'} \le C_\delta(t', t)\varepsilon/2 < \varepsilon$, so that $t' \in I_\delta$ and I_δ is open.

The upshot is that there exists $x \in [v_\delta]$ such that $\|x - y_0(\delta)\|_0 \le \varepsilon$, $0 < \delta < \delta_1$. But $y_0(\delta) = G_\delta(\mathbf{u})$ and, again by the uniqueness statement in the implicit function theorem, we get that x and u_δ differ by a shift. Hence $[u_\delta] = [v_\delta]$. $\square$

Proof of Theorem 3.7. We first prove (i) and show the existence of δ_1. Assume by contradiction that there exists a sequence $\delta_n \to 0$ and Floer trajectories $v_n \in \widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_{\delta_n}, J)$ such that J is not regular for v_n. By Proposition 4.7 we may assume, up to shifting and passing to a subsequence, that $v_n \to \mathbf{u} \in \widehat{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)$. As seen in the proof of Proposition 4.22, the limit $\mathbf{u}$ has nonconstant intermediate gradient trajectories since J is regular for $\mathbf{u}$. We can therefore apply Proposition 4.16 and get parameters $\overline{\epsilon}^n$ and vector fields ζ_n such that $v_n = \exp_{G_{\delta_n, \overline{\epsilon}^n}(\mathbf{u})}(\zeta_n)$ and $\|\zeta_n\|_{1,\delta_n,\overline{\epsilon}^n} \to 0$. By Proposition 4.18 the operator $D_{G_{\delta_n,\overline{\epsilon}^n}(\mathbf{u})}$ is surjective and admits a right inverse which is uniformly bounded with respect to δ. We infer that the operator D_{v_n} is also surjective for n large enough, a contradiction.

Let us prove (ii). Let $(\delta, v_\delta) \in \widehat{\mathcal{M}}^A_{]0,\delta_1[}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)$ and let $I(\delta) \subset]0, \delta_1[$ be a small relatively compact open interval containing δ. Since the norms $\|\cdot\|_{1,\delta'}$ are equivalent for $\delta' \in I(\delta)$, the space $\mathcal{B}_{I(\delta)} := \bigcup_{\delta' \in I(\delta)} \{\delta'\} \times \mathcal{B}_{\delta'}$ is a Banach manifold. Similarly, there is a Banach vector bundle $\mathcal{E}_{I(\delta)} \to \mathcal{B}_{I(\delta)}$ endowed with an obvious section $\bar{\partial}_{H_{I(\delta)},J}$ whose restriction to $\mathcal{B}_{\delta'}$ is $\bar{\partial}_{H_{\delta'},J}$. The restriction of its linearization $D_{(\delta,v_\delta)}$ at (δ, v_δ) to $T_{v_\delta}\mathcal{B}_\delta$ is the surjective operator D_{v_δ} of index 1, hence $D_{(\delta,v_\delta)}$ is surjective and has index 2. Therefore $\ker D_{(\delta,v_\delta)}$ projects surjectively onto $T_\delta I(\delta) = \mathbb{R}$ and the projection in (ii) is a submersion.

We now prove (iii). Let us note that, by Proposition 4.22, we have a map

$$\begin{array}{rcl} \mathcal{M}^A(p,q;H,\{f_\gamma\},J) & \to & \pi_0(\mathcal{M}^A_{]0,\delta_1[}(\overline{\gamma}_p,\underline{\gamma}_p;H,\{f_\gamma\},J)), \\ [\mathbf{u}] & \mapsto & C_{[\mathbf{u}]} := \bigcup_{\delta\in]0,\delta_1[} \{(\delta,[u_\delta])\}, \end{array}$$

where $[u_\delta]$ is the uniquely defined one-parameter family of Proposition 4.22 such that $[u_\delta] \to [\mathbf{u}]$. This map is injective because the limit of such a family $[u_\delta]$ as $\delta \to 0$ is unique. In order to prove surjectivity, let $C = \{(\delta,[v_\delta])\}$ be a connected component of $\mathcal{M}^A_{]0,\delta_1[}(\overline{\gamma}_p,\underline{\gamma}_p;H,\{f_\gamma\},J)$. By Proposition 4.7 there exists a sequence $\delta_n \to 0$ and $[\mathbf{u}] \in \mathcal{M}^A(p,q;H,\{f_\gamma\},J)$ such that $[v_{\delta_n}] \to [\mathbf{u}]$. By the uniqueness statement in Proposition 4.22 we get that $C = C_{[\mathbf{u}]}$. □

4.4 Coherent orientations

The structure of this section is as follows. We first present the construction of coherent orientations in the usual Floer setting for (H_δ, J) by adopting the point of view of [5]. We construct coherent orientations on the moduli spaces of Morse-Bott trajectories, out of which we get orientations on the space of Morse-Bott trajectories with gradient fragments. Finally, we prove Proposition 3.9.

We denote $S^1 := \mathbb{R}/\mathbb{Z}$ and, for a path of symmetric matrices $S : S^1 \to M_{2n}(\mathbb{R})$, we denote by Ψ_S the unique solution of the Cauchy problem

$$\dot{\Psi}(\theta) = J_0 S(\theta)\Psi(\theta), \quad \Psi(0) = \mathbb{1}, \quad \theta \in [0,1], \tag{85}$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}$. Then Ψ_S is a path of symplectic matrices and we denote

$$\mathcal{S} := \{S : S^1 \to M_{2n}(\mathbb{R}) \,:\, {}^tS = S \text{ and } \det(\mathbb{1} - \Psi_S(1)) \neq 0\}.$$

Let us denote by E a symplectic vector bundle of rank $2n$ over $\mathbb{C}P^1$, or $\mathbb{R}\times S^1$, or $\mathbb{C}$, with fixed trivializations in neighbourhoods of infinity in the case of $\mathbb{R}\times S^1$ and $\mathbb{C}$. We denote by

$$\mathcal{O}(\mathbb{C}P^1, E)$$

the space of linear operators $D : W^{1,p}(\mathbb{C}P^1,E) \to L^p(\mathbb{C}P^1, \Lambda^{0,1}E)$ of the form $(\partial_x + J_0\partial_y + S(z))d\bar{z}$ in a local trivialization of E, where $z = x+iy$ is a local coordinate on $\mathbb{C}P^1$. Given $\overline{S}, \underline{S} \in \mathcal{S}$ we denote by

$$\mathcal{O}(\mathbb{R}\times S^1, E; \overline{S}, \underline{S})$$

the space of linear operators $D : W^{1,p}(\mathbb{R}\times S^1, E) \to L^p(\mathbb{R}\times S^1, \Lambda^{0,1}E)$ of the form $(\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta)$ in a local trivialization of E, such that $\lim_{s\to-\infty} S(s,\cdot) = \overline{S}$ and $\lim_{s\to\infty} S(s,\cdot) = \underline{S}$ in the given trivializations of E. Given $S_0 \in \mathcal{S}$ we denote by

$$\mathcal{O}_\pm(\mathbb{C}, E; S_0)$$

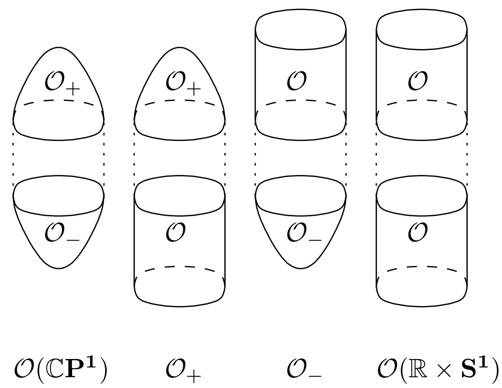

Figure 7: The four possibilities of gluing ($\mathcal{O} = \mathcal{O}(\mathbb{R}\times S^1)$).

the space of linear operators $D : W^{1,p}(\mathbb{C}, E) \to L^p(\mathbb{C}, \Lambda^{0,1}E)$ of the form $(\partial_x + J_0\partial_y + S(z))d\bar{z}$ in a local trivialization of E and such that, when expressed in holomorphic cylindrical coordinates (s, θ) with $e^{\pm 2\pi(s+i\theta)} = z$ as $(\partial_s + J_0\partial_\theta + S(s, \theta))(ds - id\theta)$, we have $\lim_{s\to\pm\infty} S(s, \theta) = S_0(\theta)$ in the given trivialization of E. Intuitively, the space $\mathcal{O}_+$ corresponds to the sphere with one positive puncture, while $\mathcal{O}_-$ corresponds to the sphere with one negative puncture.

It is a standard fact in the literature that each of the above spaces $\mathcal{O}$ is contractible and consists of Fredholm operators. Moreover, they each come equipped with a canonical real line bundle $\mathrm{Det}(\mathcal{O})$ whose fiber at D is $\mathrm{Det}(D) := (\Lambda^{\max}\ker D)\otimes(\Lambda^{\max}\mathrm{coker}\, D)^*$. Each of the bundles $\mathrm{Det}(\mathcal{O})$ is trivial since the base is contractible.

We now define gluing operations between elements of the above spaces (see Figure 7). Let $K \in \mathcal{O}_+(\mathbb{C}, E; S_0)$ or $K \in \mathcal{O}(\mathbb{R}\times S^1, E; \overline{S}, S_0)$, and $L \in \mathcal{O}_-(\mathbb{C}, F; S_0)$ or $L \in \mathcal{O}(\mathbb{R}\times S^1, F; S_0, \underline{S})$. Let us choose a cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(s) = 0$ if $s \le 0$ and $\beta(s) = 1$ if $s \ge 1$. Given $R > 0$ large we define operators K_R and L_R by replacing S in the asymptotic expressions of K and L by $S_0+\beta(R-s)(S-S_0)$ and $S_0+\beta(R+s)(S-S_0)$ respectively. We cut out semi-infinite cylinders $\{s > R\}$ from the base of E, $\{s < -R\}$ from the base of F, then identify their boundaries using the coordinate θ. We glue the vector bundles E and F using their given trivializations near infinity and denote the resulting vector bundle by $E\#F$. We define $K\#_R L$ by concatenating K_R and L_R, so that $K\#_R L$ belongs to one of the spaces $\mathcal{O}(\mathbb{C}P^1, E\#F)$, $\mathcal{O}_+(\mathbb{C}, E\#F; \underline{S})$, $\mathcal{O}_-(\mathbb{C}, E\#F; \overline{S})$, or $\mathcal{O}(\mathbb{R}\times S^1, E\#F; \overline{S}, \underline{S})$.

Following [5, Corollary 7], for R large enough there is a natural isomorphism $\mathrm{Det}(K)\otimes\mathrm{Det}(L) \xrightarrow{\sim} \mathrm{Det}(K\#_R L)$ defined up to homotopy. In particular, given orientations o_K of $\mathrm{Det}(K)$ and o_L of $\mathrm{Det}(L)$, we induce a canonical orientation $o_K\#o_L$ of $\mathrm{Det}(K\#_R L)$. Moreover, this operation on orientations is associative [13, Theorem 10].

We describe now, following [5], a procedure for constructing orientations on

the spaces $\mathcal{O}(\mathbb{R} \times S^1, E; \overline{S}, \underline{S})$ which are coherent with respect to the gluing operation, in the sense of [13, Definition 11]. We denote by θ_n a trivial symplectic vector bundle of rank $2n$. We first note that each determinant bundle $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1, E))$ is naturally oriented since $\mathcal{O}(\mathbb{C}P^1, E)$ contains the connected space of complex linear operators and the latter have kernels and cokernels which are canonically oriented as complex vector spaces. We now choose arbitrary orientations of the determinant bundles $\mathrm{Det}(\mathcal{O}_+(\mathbb{C}, E; S_0))$ such that the trivialization of E at infinity extends to $\mathbb{C}$.

Remark 4.24. Note that, if S_0 commutes with J_0, the set of $\mathbb{C}$-linear operators in $\mathcal{O}_+(\mathbb{C}, E; S_0)$ forms a nonempty convex set, hence $\mathrm{Det}(\mathcal{O}_+(\mathbb{C}, E; S_0))$ has a canonical orientation.

We induce orientations on the determinant bundles $\mathrm{Det}(\mathcal{O}_-(\mathbb{C}, E; S_0))$ such that the trivialization of E at infinity extends to $\mathbb{C}$ by requiring that the orientation induced by gluing on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1, \theta_n))$ is the canonical one. Finally, we induce orientations on $\mathrm{Det}(\mathcal{O}(\mathbb{R}\times S^1, E; \overline{S}, \underline{S}))$ by requiring that the orientation induced on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1, \theta_n \# E \# \theta_n))$ by the gluing operation

$$\mathcal{O}_+(\mathbb{C}, \theta_n; \overline{S}) \times \mathcal{O}(\mathbb{R} \times S^1, E; \overline{S}, \underline{S}) \times \mathcal{O}_-(\mathbb{C}, \theta_n; \underline{S}) \to \mathcal{O}(\mathbb{C}P^1, \theta_n \# E \# \theta_n)$$

is the canonical one. It is proved in [5] that this defines a system of coherent orientations.

The general procedure for inducing orientations of the spaces of Floer trajectories $\widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ out of a system of coherent orientations goes as follows. Let $\overline{\Psi}_p$, $\underline{\Psi}_q$ denote the linearizations of the Hamiltonian flow of H_δ along $\overline{\gamma}_p$, $\underline{\gamma}_q$ in their fixed respective trivializations and let $\overline{S}_p, \underline{S}_q \in \mathcal{S}$ be the corresponding paths of symmetric matrices as in (85). Let E be a symplectic vector bundle over $\mathbb{R} \times S^1$ with fixed trivializations at infinity and relative first Chern class equal to $\langle c_1(T\widehat{W}), A\rangle$. For each $u \in \widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$ there is an isomorphism of symplectic vector bundles $\Phi_u : u^*T\widehat{W} \xrightarrow{\sim} E$, chosen to depend continuously on u. There is a map

$$\widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J) \to \mathcal{O}(\mathbb{R} \times S^1, E; \overline{S}_p, \underline{S}_q), \quad u \mapsto \Phi_u \circ \widetilde{D}_u \circ \Phi_u^{-1},$$

where $\widetilde{D}_u$ has the same analytical expression as the linearized operator $D_u : W^{1,p}(\mathbb{R}\times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta) \oplus \overline{V}_u \oplus \underline{V}_u \to L^p(\mathbb{R}\times S^1, u^*T\widehat{W}; e^{d|s|}ds\, d\theta)$ considered in Section 4.3. Under the assumption that J is regular and because of elliptic regularity, the operators $\widetilde{D}_u$ and D_u have the same kernel, consisting of smooth elements. Hence their determinant lines are naturally isomorphic. It follows that the pull-back of $\mathrm{Det}(\mathcal{O})$ under the above map is naturally isomorphic to $\Lambda^{\max} T\widehat{\mathcal{M}} = \Lambda^{\max} \ker D_u$, and we get an orientation on $\widehat{\mathcal{M}}$.

If the dimension is one, the space $\widehat{\mathcal{M}}$ has a canonical orientation given at each point u by the vector field $\partial_s u$. Comparing this with the orientation constructed above associates to each connected component $[u]$ of $\widehat{\mathcal{M}}$ a sign $\epsilon(u)$.

Lemma 4.25. *Let $S_1 \in \mathcal{S}$ and define $S_m(\theta) := S_1(m\theta)$. Assume $S_m \in \mathcal{S}$ and define an automorphism ϕ_m of $\mathcal{O}_+(\mathbb{C}, E; S_m)$ by conjugation with the map $z \mapsto e^{2i\pi/m}z$. Then ϕ_m is orientation reversing for* $\mathrm{Det}(\mathcal{O}_+(\mathbb{C}, E; S_m))$ *if and only if m is even and the difference of Conley-Zehnder indices $\mu_{CZ}(\Psi_{S_m}) - \mu_{CZ}(\Psi_{S_1})$ is odd.*

Proof. We start by explaining how ϕ_m acts on the orientations of the determinant bundle. The operators $K \in \mathcal{O}_+(\mathbb{C}, E; S_m)$ which are invariant under conjugation by ϕ_m, i.e. $K(\zeta \circ \phi_m) \circ \phi_m^{-1} = K(\zeta)$ for all ζ, form a convex and, in particular, connected set. Since ϕ_m acts on $\ker K$ and $\operatorname{coker} K$, it also acts on $\mathrm{Det}(K)$ and the induced action on orientations extends to $\mathrm{Det}(\mathcal{O}_+(\mathbb{C}, E; S_m))$.

There is a bijective correspondence between operators $K_1 \in \mathcal{O}_+(\mathbb{C}, E; S_1)$ and operators $K \in \mathcal{O}_+(\mathbb{C}, E; S_m)$ which are invariant under conjugation by ϕ_m, in which case the pull-back of $\ker K_1$ under $z \mapsto z^m$ is the 1-eigenspace of ϕ_m acting on $\ker K$. Since $\ker K$ splits as a direct sum of eigenspaces corresponding to the m-th roots of unity and since imaginary roots give rise to even-dimensional eigenspaces, we infer that the dimension of the -1-eigenspace has the parity of $\dim\ker K - \dim\ker K_1$. This fact is relevant in our situation since ϕ_m reverses the orientation of $\ker K$ if and only if this dimension is odd. Similarly, ϕ_m reverses the orientation of $\operatorname{coker} K$ if and only if $\dim\operatorname{coker} K - \dim\operatorname{coker} K_1$ is odd. As a conclusion, ϕ_m reverses the orientation of $\mathrm{Det}(K)$ if and only if $\mathrm{ind}(K) - \mathrm{ind}(K_1) = \mu_{CZ}(\Psi_{S_m}) - \mu_{CZ}(\Psi_{S_1})$ is odd. This can happen of course only if -1 is an m-th root of unity, i.e. m is even. □

Remark 4.26. The proof of Lemma 4.25 shows that, if m is odd, the difference of Conley-Zehnder indices is automatically even.

Lemma 4.27. *Let $S_1 \in \mathcal{S}$, define $S_m(\theta) := S_1(m\theta)$ and assume $S_m \in \mathcal{S}$. Let $T \in \mathcal{O}(\mathbb{R} \times S^1, \theta_n; S_m, S_m)$ be an element of the form*

$$T := \partial_s + J_0\partial_\theta + S_m(\theta - \beta(s)/m),$$

with $\beta : \mathbb{R} \to [0, 1]$ a smooth function satisfying $\beta(s) = 0$ near $-\infty$, $\beta(s) = 1$ near $+\infty$ and with derivative uniformly bounded by some small constant c. We denote by $\mathcal{O}$ one of the spaces $\mathcal{O}_+(\mathbb{C}, E; S_m)$ or $\mathcal{O}(\mathbb{R} \times S^1, E; S, S_m)$, $S \in \mathcal{S}$. The family $\psi = \{\psi_R\}$, $R > 0$ of automorphisms of $\mathcal{O}$ defined by

$$\psi_R(D) := D\#_R T$$

induces an action on the orientations of $\mathrm{Det}(\mathcal{O})$ *which is reversing if and only if m is even and the difference of Conley-Zehnder indices $\mu_{CZ}(\Psi_{S_m}) - \mu_{CZ}(\Psi_{S_1})$ is odd.*

Proof. Note that T is an isomorphism if c is small enough, by the same argument as the one for D''_u in the proof of Proposition 4.9.

We now explain what is the action of ψ on the orientations of $\mathrm{Det}(\mathcal{O})$. Let $D \in \mathcal{O}$ and let $V \subset L^p$ be a finite dimensional vector space spanned by

smooth sections with compact support, such that $V + \operatorname{im} D = L^p$. We define the stabilization of D by V as

$$D^V : V \oplus W^{1,p} \to L^p, \qquad (v, \zeta) \mapsto v + D\zeta.$$

Then D^V is a surjective Fredholm operator and there is a canonical isomorphism $\mathrm{Det}(D) \simeq \Lambda^{\max} \ker D^V \otimes \Lambda^{\max} V^*$. For R large enough the glued operator $D_R = D^V \#_R T : V \oplus W^{1,p} \to L^p$ is surjective with a uniformly bounded right inverse Q_R, and moreover the projection onto $\ker D_R$ given by $\mathbb{1} - Q_R D_R$ is an isomorphism when restricted to $\ker D^V$ (see [5, Corollary 6], as well as [13, Proposition 9] for a slightly different setup). Since $D^V \#_R T = (D \#_R T)^V$, this induces a natural isomorphism between $\mathrm{Det}(D)$ and $\mathrm{Det}(\psi_R(D))$.

The gluing of orientations is associative, hence it is enough to prove the statement for $\mathcal{O} = \mathcal{O}_+(\mathbb{C}, \theta_n; S_m)$. We claim that the action induced by ψ is the same as the one induced by ϕ_m in Lemma 4.25. Let us choose $D \in \mathcal{O}$ which is s-independent for s large enough, and let D^V be a surjective stabilization. We construct a continuous path in $\mathcal{O}$ from $\psi_R(D^V) := \psi_R(D)^V$ to $\phi_m(D^V) := \phi_m(D)^V$ as follows. Let D_t^V be the conjugation of D^V by $r_t : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^{-2i\pi t/m} z$, and let T_t be the operator $\partial_s + J_0 \partial_\theta + S_m(\theta - (t + (1-t)\beta(s))/m)$. Then $D_t^V \#_R T_t$ interpolates between $\psi_R(D^V)$ and $\phi_m(D^V)$ as t varies from 0 to 1. This is a path of surjective operators admitting a continuous family of right inverses Q_t. Given a basis $(\zeta_1, \ldots, \zeta_k)$ of $\ker D^V$, a basis of $\ker D_t^V$ is given by $(\zeta_1, \ldots, \zeta_k) \circ r_t$. By projecting along $\operatorname{im} Q_t$ we obtain a basis of $\ker D_t^V \#_R T_t$. For $t = 1$, since $D_1^V = D_1^V \#_R T_1$, the elements $\zeta_i \circ r_1$ are preserved by the projection and form a basis of $\phi_m^{-1}(D^V)$ which is exactly the one giving the action of ϕ_m^{-1} (or ϕ_m) on orientations, as explained in Lemma 4.25. □

Lemma 4.28. *Let $\gamma \in \mathcal{P}_\lambda^{\leq \alpha}$ and γ_p, γ_q be the orbits corresponding to the minimum p and maximum q of f_γ respectively. For $\delta > 0$ small enough, the moduli space $\mathcal{M}^A(\gamma_p, \gamma_q; H_\delta, J)$ is empty if $A \neq 0$, while for $A = 0$ it consists of exactly two elements u_1, u_2 corresponding to the two gradient trajectories of f_γ running from p to q. Moreover, they satisfy*

$$\epsilon(u_1) + \epsilon(u_2) = \begin{cases} 0, & \text{if } \gamma \text{ is a good orbit,} \\ \pm 2, & \text{if } \gamma \text{ is a bad orbit.} \end{cases}$$

Proof. Let c_1, c_2 be the gradient trajectories of f_γ running from p to q. By Theorem 3.7, for $\delta > 0$ small enough each element $[u_\delta] \in \mathcal{M}^A(\gamma_p, \gamma_q; H_\delta, J)$ corresponds to a unique Floer trajectory with gradient fragments $[\mathbf{u}]$ whose endpoints are p and q. For energy reasons there can be no nonconstant Floer trajectory involved in $[\mathbf{u}]$ and therefore $[\mathbf{u}]$ is either c_1 or c_2. Since the cylinders u_1 and u_2 of the form $u_{\delta, \overline{\gamma}, -\infty, +\infty}$ associated to c_1 and c_2 are already Floer trajectories for H_δ, we infer that $[u_\delta]$ equals either $[u_1]$ or $[u_2]$, and the homology class A is necessarily zero. Let us introduce the notation $\epsilon(\gamma) := 1$ if γ is a good orbit and $\epsilon(\gamma) := -1$ if γ is a bad orbit. The conclusion of the Lemma is equivalent to the relation

$$\epsilon(u_1) = -\epsilon(\gamma) \epsilon(u_2). \tag{86}$$

Let us choose a symplectic trivialization $\Phi_\gamma : T\widehat{W}|_{S_\gamma} \to S_\gamma \times (\mathbb{R}\times\mathbb{R}^{2n-1})$ such that $\Phi_\gamma(X_H) = (1,0)$. We assume without loss of generality that $\dot{c}_1$ is a positive multiple of X_H, so that $\Phi_\gamma(\partial_s u_1) = (f_1, 0)$ with $f_1 > 0$ and $\Phi_\gamma(\partial_s u_2) = (f_2, 0)$ with $f_2 < 0$. We denote by $\overline{D}_{u_1}$, $\overline{D}_{u_2}$ the elements of $\mathcal{O}(\mathbb{R}\times S^1, \theta_n; S_p, S_q)$ obtained by conjugation of $\widetilde{D}_{u_1}$, $\widetilde{D}_{u_2}$ with Φ_γ. The main point is to consider the operator $\psi(\overline{D}_{u_1}) = \overline{D}_{u_1}\#_R T$, with T as in Lemma 4.27. A basis of $\mathrm{Det}(\overline{D}_{u_i})$ corresponding to the coherent orientation is by definition $\epsilon(u_i)(f_i, 0)$, $i = 1, 2$. The image of this basis under the action of ψ is given by $\epsilon(u_1)(f_1^{\#}, 0)$, for some $f_1^{\#} \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R})$ with $\|f_1^{\#} - f_1\|_{1,p}$ arbitrarily small with $R \to \infty$, hence $f_1^{\#} > 0$ for R large enough. By Lemma 4.27, a basis of $\mathrm{Det}(\overline{D}_{u_1}\#_R T)$ corresponding to the coherent orientation is $\epsilon(\gamma)\epsilon(u_1)(f_1^{\#}, 0)$. Finally, the operators $\overline{D}_{u_1}\#_R T$ and $\overline{D}_{u_2}$ can be connected by a continuous path of operators D_t, $t \in [0,1]$ satisfying properties (ii)-(iv) in the proof of Proposition 4.9, as well as the following weaker form of property (i) therein.

(i') there exists a smooth path $c : \mathbb{R} \to S_\gamma$ with $c(\pm\infty)$ being fixed critical points of f_γ, such that $\|S(s,\theta) - \overline{S}(\theta + \vartheta\circ c(s) - \vartheta\circ c(-\infty))\|$ is bounded by a constant multiple of δ.

The connected components of the set of operators satisfying (i') and (ii)-(iv) are indexed by homotopy classes of paths c as above. Gluing $\overline{D}_{u_1}$ to T has precisely the effect of concatenating c_1 with $(\gamma_q|_{[0,1/m]})^{-1}$, which is homotopic to c_2. The proof of Proposition 4.9 works the same with the weaker assumption (i') and shows that the operators D_t are surjective and that ker D_t is generated by an element of the form $(f_t, 0)$, where $f_t \in W^{1,p}(\mathbb{R}\times S^1, \mathbb{R})$ has constant sign for $t \in [0,1]$. We conclude that $\epsilon(u_1)\epsilon(\gamma)f_1^{\#}$ and $\epsilon(u_2)f_2$ have the same sign, hence (86) is proved. □

We now generalize the construction of coherent orientations to the moduli spaces of Morse-Bott trajectories with gradient fragments. We define $\widetilde{\mathcal{S}}$ to be the space of loops of symmetric matrices $S : S^1 \to M_{2n}(\mathbb{R})$ such that the symplectic matrix $\Psi_S(1)$ defined by (85) has exactly one eigenvalue equal to 1, corresponding to the eigenspace $\mathbb{R}\oplus 0 \subset \mathbb{R}\oplus\mathbb{R}^{2n-1} = \mathbb{R}^{2n}$. Let $\beta : \mathbb{R} \to [0,1]$ be a smooth function equal to 0 near $-\infty$ and equal to 1 near $+\infty$. We define $\overline{V}$, $\underline{V}$ to be the one-dimensional real vector spaces generated by the vector-valued functions $(1-\beta(s))(1,0)$ and $\beta(s)(1,0)$ respectively. In the following we denote by $W^{1,p,d} = W^{1,p}(e^{d|s|}ds\,d\theta)$, $L^{p,d} = L^p(e^{d|s|}ds\,d\theta)$. Given $\overline{S}, \underline{S} \in \widetilde{\mathcal{S}}$ we denote by

$$\widetilde{\mathcal{O}}(\mathbb{R}\times S^1, E; \overline{S}, \underline{S})$$

the space of linear operators $D : W^{1,p,d}(\mathbb{R}\times S^1, E) \oplus \overline{V} \oplus \underline{V} \to L^{p,d}(\mathbb{R}\times S^1, \Lambda^{0,1}E)$ of the form $(\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta)$ in a local trivialization of E, for which there exist $\overline{\theta}, \underline{\theta} \in \mathbb{R}/\mathbb{Z}$ such that $\lim_{s\to-\infty} S(s,\theta) = \overline{S}(\theta + \overline{\theta})$ and $\lim_{s\to\infty} S(s,\theta) = \underline{S}(\theta + \underline{\theta})$ in the given trivializations at infinity of E. Given $\overline{S}_0 \in \mathcal{S}$, $\underline{S} \in \widetilde{\mathcal{S}}$ we denote by

$$\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1, E; \overline{S}_0, \underline{S})$$

the space of linear operators

$$D : W^{1,p}(\mathbb{R}\times S^1, E; g_+(s)ds\,d\theta) \oplus \underline{V} \to L^p(\mathbb{R}\times S^1, \Lambda^{0,1}E; g_+(s)ds\,d\theta)$$

with $g_+(s) := \max(1, e^{ds})$, which are of the form $(\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta)$ in a local trivialization of E, and for which there exists $\underline{\theta} \in \mathbb{R}/\mathbb{Z}$ such that $\lim_{s\to-\infty} S(s,\theta) = \overline{S}_0(\theta)$ and $\lim_{s\to\infty} S(s,\theta) = \underline{S}(\theta + \underline{\theta})$ in the given trivializations at infinity of E. Given $\overline{S} \in \widetilde{\mathcal{S}}$, $\underline{S}_0 \in \mathcal{S}$ we denote by

$$\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, E; \overline{S}, \underline{S}_0)$$

the space of linear operators

$$D : W^{1,p}(\mathbb{R}\times S^1, E; g_-(s)ds\,d\theta) \oplus \overline{V} \to L^p(\mathbb{R}\times S^1, \Lambda^{0,1}E; g_-(s)ds\,d\theta)$$

with $g_-(s) := \max(1, e^{-ds})$, which are of the form $(\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta)$ in a local trivialization of E, and for which there exists $\overline{\theta} \in \mathbb{R}/\mathbb{Z}$ such that $\lim_{s\to-\infty} S(s,\theta) = \overline{S}(\theta + \overline{\theta})$ and $\lim_{s\to\infty} S(s,\theta) = \underline{S}_0(\theta)$ in the given trivializations at infinity of E. Given $\widetilde{S} \in \widetilde{\mathcal{S}}$ we denote by

$$\widetilde{\mathcal{O}}_\pm(\mathbb{C}, E; \widetilde{S})$$

the space of linear operators $D : W^{1,p,d}(\mathbb{C}, E) \oplus V_\pm \to L^{p,d}(\mathbb{C}, \Lambda^{0,1}E)$ of the form $(\partial_x + J_0\partial_y + S(z))d\bar{z}$ in a local trivialization of E and such that, when expressed in holomorphic cylindrical coordinates (s,θ) with $e^{\pm 2\pi(s+i\theta)} = z$ as $(\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta)$, there exists $\theta_\pm \in \mathbb{R}/\mathbb{Z}$ so that $\lim_{s\to\pm\infty} S(s,\theta) = \widetilde{S}(\theta + \theta_\pm)$ in the given trivialization of E near infinity. Here we use the notation $V_+ := \overline{V}$ and $V_- := \underline{V}$.

Due to the exponential weights, each of the above spaces $\widetilde{\mathcal{O}}$ consists of Fredholm operators and comes equipped with a canonical real line bundle $\mathrm{Det}(\widetilde{\mathcal{O}})$ whose fiber at D is $\mathrm{Det}(D)$. Unlike in the nondegenerate case, the spaces $\widetilde{\mathcal{O}}$ are not generally contractible, hence we have to investigate the orientability of $\mathrm{Det}(\widetilde{\mathcal{O}})$.

Given $S \in \widetilde{\mathcal{S}}$ we define $m = m(S)$ to be the maximal positive integer such that $S(\theta + 1/m) = S(\theta)$, $\theta \in \mathbb{R}/\mathbb{Z}$. The number m is infinite if and only if the loop S is constant, in which case the spaces $\widetilde{\mathcal{O}}_\pm(\mathbb{C}, E; S)$, $\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1, E; \overline{S}_0, S)$, $\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, E; S, \underline{S}_0)$ are contractible. In the following we shall restrict ourselves to nonconstant loops $S \in \widetilde{\mathcal{S}}$, in which case the above spaces have the homotopy type of S^1, while $\widetilde{\mathcal{O}}(\mathbb{R}\times S^1, E; \overline{S}, \underline{S})$ has the homotopy type of $S^1 \times S^1$ (this is because they fiber over S^1, respectively $S^1 \times S^1$ with contractible fibers). We denote by $S_1 \in \widetilde{\mathcal{S}}$ the unique loop such that $S(\theta) = S_1(m\theta)$.

Lemma 4.29. *Let $S \in \widetilde{\mathcal{S}}$ be nonconstant. Then* $\mathrm{Det}(\widetilde{\mathcal{O}}_\pm(\mathbb{C}, E; S))$ *is nonorientable if and only if m is even and $\mu_{RS}(S) - \mu_{RS}(S_1)$ is odd.*

Proof. We prove the statement only for $\widetilde{\mathcal{O}}_+ := \widetilde{\mathcal{O}}_\pm(\mathbb{C}, E; S)$, the proof of the other case being similar. The following two remarks will allow us to apply

Lemma 4.25. First, $\mathrm{Det}(D)$ is naturally isomorphic to $\mathrm{Det}(D|_{W^{1,p,d}}) \otimes V_+$ and V_+ is a trivial bundle over $\widetilde{\mathcal{O}}_+$. Second, the operator $D|_{W^{1,p,d}}$ is conjugated to an operator $\widetilde{D} \in \mathcal{O}_+(\mathbb{C}, E; S - \frac{d}{p}\mathbb{1})$. Hence it is enough to study the orientability of the bundle $\widetilde{\mathrm{Det}}(\widetilde{\mathcal{O}}_+)$ over $\widetilde{\mathcal{O}}_+$ with fiber $\mathrm{Det}(\widetilde{D})$.

The bundle $\widetilde{\mathrm{Det}}(\widetilde{\mathcal{O}}_+)$ is orientable if and only if its restriction to a loop generating $\pi_1(\widetilde{\mathcal{O}})$ is orientable. After choosing $D \in \widetilde{\mathcal{O}}_+$ which is invariant under conjugation with $z \mapsto e^{-2i\pi/m}$, the conjugation of D by $r_t : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^{-2i\pi t/m} z$ provides such a loop D_t, $t \in [0,1]$ with $D_0 = D_1 = D$. The orientation on $\mathrm{Det}(\widetilde{D}_1)$ obtained by continuation along the path D_t from an orientation on $\mathrm{Det}(\widetilde{D}_0)$ is the same as the one induced by the action of ϕ_m^{-1} (or ϕ_m) in Lemma 4.25. Since $\mu_{CZ}(S - \frac{d}{p}\mathbb{1}) = \mu_{RS}(S) - 1/2$ and $\mu_{CZ}(S_1 - \frac{d}{p}\mathbb{1}) = \mu_{RS}(S_1) - 1/2$, the statement follows from Lemma 4.25. □

The same kind of argument gives the following result.

Lemma 4.30. *Let $S, \overline{S}, \underline{S} \in \widetilde{\mathcal{S}}$ be nonconstant and $\overline{S}_0, \underline{S}_0 \in \mathcal{S}$. The line bundles $\mathrm{Det}(\widetilde{\mathcal{O}}(\mathbb{R} \times S^1, E; \overline{S}, \underline{S}))$, $\mathrm{Det}(\widetilde{\mathcal{O}}^u(\mathbb{R} \times S^1, E; \overline{S}_0, S))$, $\mathrm{Det}(\widetilde{\mathcal{O}}^s(\mathbb{R} \times S^1, E; S, \underline{S}_0))$ are nonorientable if and only if the condition in Lemma 4.29 holds for S and for one of $\overline{S}, \underline{S}$.* □

The previous results motivate the following definition. We denote

$$\widetilde{\mathcal{S}}_{\mathrm{good}} := \{S \in \widetilde{\mathcal{S}} \,:\, S \text{ constant or } \mu_{RS}(S) - \mu_{RS}(S_1) \text{ is even}\},$$

and

$$\widetilde{\mathcal{S}}_{\mathrm{bad}} := \{S \in \widetilde{\mathcal{S}} \,:\, S \text{ nonconstant and } \mu_{RS}(S) - \mu_{RS}(S_1) \text{ is odd}\},$$

so that $\widetilde{\mathcal{S}}_{\mathrm{bad}} = \widetilde{\mathcal{S}} \setminus \widetilde{\mathcal{S}}_{\mathrm{good}}$. Although the determinant lines over the various spaces $\widetilde{\mathcal{O}}$ are nonorientable if one of the asymptotes is in $\widetilde{\mathcal{S}}_{\mathrm{bad}}$, we can construct **covers $\widetilde{\widetilde{\mathcal{O}}}$ of $\widetilde{\mathcal{O}}$** over which the determinant lines become orientable. Let $\underline{S}_0 \in \mathcal{S}$ and $\overline{S} \in \widetilde{\mathcal{S}}_{\mathrm{bad}}$ with $\overline{S}(\theta) = \overline{S}_1(m\theta)$, $\theta \in \mathbb{R}/\mathbb{Z}$. We define $\widetilde{\widetilde{\mathcal{O}}}^s(\mathbb{R} \times S^1, E; \overline{S}, \underline{S}_0)$ to consist of pairs $(D, \overline{\theta})$ such that $\overline{\theta} \in \mathbb{R}/\frac{2}{m}\mathbb{Z}$, $D = (\partial_s + J_0\partial_\theta + S(s,\theta))(ds - id\theta) \in \widetilde{\mathcal{O}}^s(\mathbb{R} \times S^1, E; \overline{S}, \underline{S}_0)$ with $\lim_{s \to -\infty} S(s,\theta) = \overline{S}(\theta + \overline{\theta})$. The obvious projection is $\widetilde{\widetilde{\mathcal{O}}}^s \to \widetilde{\mathcal{O}}^s$ is a double cover and the lift of the determinant bundle to $\widetilde{\widetilde{\mathcal{O}}}^s$ is orientable. We define in a completely analogous manner double covers $\widetilde{\widetilde{\mathcal{O}}}_\pm(S) \to \widetilde{\mathcal{O}}_\pm(S)$, $\widetilde{\widetilde{\mathcal{O}}}^u(\overline{S}_0, S) \to \widetilde{\mathcal{O}}^u(\overline{S}_0, S)$, $S \in \widetilde{\mathcal{S}}_{\mathrm{bad}}$ and a cover $\widetilde{\widetilde{\mathcal{O}}}(\overline{S}, \underline{S}) \to \widetilde{\mathcal{O}}(\overline{S}, \underline{S})$ which is double if exactly one of $\overline{S}, \underline{S}$ is in $\widetilde{\mathcal{S}}_{\mathrm{bad}}$, and quadruple if both $\overline{S}, \underline{S}$ are in $\widetilde{\mathcal{S}}_{\mathrm{bad}}$.

We define now gluing operations between elements of the various spaces $\widetilde{\mathcal{O}}$. Let K in $\widetilde{\mathcal{O}}_+(\mathbb{C}, E; S)$, $\widetilde{\mathcal{O}}(\mathbb{R} \times S^1, E; \overline{S}, S)$, or $\widetilde{\mathcal{O}}^u(\mathbb{R} \times S^1, E; \overline{S}_0, S)$, and L in $\widetilde{\mathcal{O}}_-(\mathbb{C}, F; S)$, $\widetilde{\mathcal{O}}(\mathbb{R} \times S^1, F; S, \underline{S})$, or $\widetilde{\mathcal{O}}^s(\mathbb{R} \times S^1, F; S, \underline{S}_0)$. We denote by S_K, respectively S_L the matrix valued functions involved in K and L near infinity. We assume that

$$\lim_{s \to +\infty} S_K(s, \cdot) = \lim_{s \to -\infty} S_L(s, \cdot) = S(\cdot + \theta_0) =: S_{\theta_0}$$

for some $\theta_0 \in \mathbb{R}/\mathbb{Z}$. We choose a cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta(s) = 0$ if $s \leq 0$ and $\beta(s) = 1$ if $s \geq 1$. Given $R > 0$ large we define operators K_R and L_R by replacing S_K and S_L by $S_{\theta_0} + \beta(R-s)(S_K - S_{\theta_0})$ and $S_{\theta_0} + \beta(R+s)(S_L - S_{\theta_0})$ respectively. We cut out semi-infinite cylinders $\{s > R\}$ from the base of E, $\{s < -R\}$ from the base of F, then identify their boundaries using the coordinate θ. We glue the vector bundles E and F using their given trivializations near infinity and denote the resulting vector bundle by $E\#F$. We define $K\#_R L$ by concatenating K_R and L_R, so that $K\#_R L$ belongs to one of the spaces $\mathcal{O}(\mathbb{C}P^1)$, $\widetilde{\mathcal{O}}_+(\mathbb{C};\underline{S})$, $\mathcal{O}_+(\mathbb{C};\underline{S}_0)$, or $\widetilde{\mathcal{O}}_-(\mathbb{R}\times S^1;\overline{S})$, $\widetilde{\mathcal{O}}(\mathbb{R}\times S^1;\overline{S},\underline{S})$, $\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1;\overline{S},\underline{S}_0)$, or $\mathcal{O}_-(\mathbb{C};\overline{S}_0)$, $\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1;\overline{S}_0,\underline{S})$, $\mathcal{O}(\mathbb{R}\times S^1;\overline{S}_0,\underline{S}_0)$, where we have omitted the symbol $E\#F$ from the notation.

The above gluing operations admit a straightforward extension to the spaces $\widetilde{\widetilde{\mathcal{O}}}$. For example, two elements $(K,\underline{\theta}) \in \widetilde{\widetilde{\mathcal{O}}}^u(\overline{S}_0, S)$, $(L,\overline{\theta}) \in \widetilde{\widetilde{\mathcal{O}}}^s(S,\underline{S}_0)$ can be glued if $\underline{\theta} = \overline{\theta}$, in which case they give rise to an element $K\#_R L \in \mathcal{O}(\overline{S}_0,\underline{S}_0)$.

Recall that the domain of an operator D in some $\widetilde{\mathcal{O}}$ contains a canonically oriented 1-dimensional summand for each asymptote in $\widetilde{\mathcal{S}}$, together with a canonical isomorphism with $\mathbb{R}$. We denote by V_K, V_L the summands corresponding to the asymptote S of K and L respectively, and we let $V := V_K \oplus_{\mathbb{R}} V_L$ be their (canonically oriented) fibered sum. By [5, Corollary 6], for $R > 0$ large enough there is a natural isomorphism $\mathrm{Det}(K \oplus_{\mathbb{R}} L) \simeq \mathrm{Det}(K\#_R L)$ defined up to homotopy, where $K \oplus_{\mathbb{R}} L$ is the restriction of $K \oplus L$ to the fibered sum of their domains. Since V is canonically oriented, it follows that $\mathrm{Det}(K \oplus_{\mathbb{R}} L)$ is canonically isomorphic to $\mathrm{Det}(K \oplus L) \simeq \mathrm{Det}(K) \otimes \mathrm{Det}(L)$. Hence we obtain a canonical isomorphism $\mathrm{Det}(K) \otimes \mathrm{Det}(L) \xrightarrow{\sim} \mathrm{Det}(K\#_R L)$ defined up to homotopy, and inducing an associative gluing operation for orientations. Similar considerations apply to the elements of the spaces $\widetilde{\widetilde{\mathcal{O}}}$.

Remark 4.31. We can construct a system of coherent orientations on the determinant line bundles $\mathrm{Det}(\widetilde{\mathcal{O}}_\pm(\mathbb{C}, E; S))$ and $\mathrm{Det}(\widetilde{\mathcal{O}}(\mathbb{R}\times S^1, E; \overline{S}, \underline{S}))$ with $S, \overline{S}, \underline{S} \in \widetilde{\mathcal{S}}_{\mathrm{good}}$ by the same procedure as for the spaces $\mathcal{O}$. We can moreover extend this to a system of coherent orientations involving all spaces $\mathcal{O}$, $\widetilde{\mathcal{O}}$ and $\widetilde{\widetilde{\mathcal{O}}}$. Nevertheless, if we want that certain orientations have a geometric meaning, we have to impose compatibility conditions which seem ad-hoc in such a general setup. This is why we restrict ourselves in the sequel to the spaces $\mathcal{O}$, $\widetilde{\mathcal{O}}$ and $\widetilde{\widetilde{\mathcal{O}}}$ which are relevant for our geometric situation.

We use now the notations of Section 3. Given $\gamma \in \mathcal{P}(H)$ we denote by Ψ_γ the linearization of the Hamiltonian flow along γ given by (21) and let $S_\gamma : \mathbb{R}/\mathbb{Z} \to M_{2n}(\mathbb{R})$ be the corresponding loop of symmetric matrices defined by $\dot{\Psi}_\gamma = J_0 S_\gamma \Psi_\gamma$. Then $S_\gamma \in \widetilde{\mathcal{S}}_{\mathrm{good}}$ if and only if γ is a good orbit. We similarly define S_{γ_q} for each $\gamma_q \in \mathcal{P}(H_\delta)$, with $q \in \mathrm{Crit}(f_\gamma)$. For $\overline{\gamma} \in \mathcal{P}(H)$, $\underline{\gamma}_q \in \mathcal{P}(H_\delta)$ we denote $\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, E; \overline{\gamma}, \underline{\gamma}_q) := \widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, E; S_{\overline{\gamma}}, S_{\underline{\gamma}_q})$ etc.

Convention. In what follows the spaces $\widetilde{\mathcal{O}}$ will be understood to be indexed

only by good orbits, whereas if one of the asymptotic orbits is bad we use the corresponding double or quadruple cover $\widetilde{\widetilde{\mathcal{O}}}$.

We construct orientations on the determinant bundles over all spaces $\mathcal{O}$, $\widetilde{\mathcal{O}}$, $\widetilde{\widetilde{\mathcal{O}}}$ indexed by the elements of $\mathcal{P}(H)$ and $\mathcal{P}(H_\delta)$ as follows. We start by choosing arbitrary orientations of $\mathrm{Det}(\widetilde{\mathcal{O}}_+(\mathbb{C},E;\gamma))$, respectively $\mathrm{Det}(\widetilde{\widetilde{\mathcal{O}}}_+(\mathbb{C},E;\gamma))$, $\gamma \in \mathcal{P}(H)$ such that the trivialization of E at infinity extends to $\mathbb{C}$. We then choose orientations of $\mathrm{Det}(\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1,E;\gamma,\gamma_q))$, respectively $\mathrm{Det}(\widetilde{\widetilde{\mathcal{O}}}^s(\mathbb{R}\times S^1,E;\gamma,\gamma_q))$, $\gamma \in \mathcal{P}(H)$, $q \in \mathrm{Crit}(f_\gamma)$ such that the trivializations of E at infinity extend to $\mathbb{R}\times S^1$, as follows. If γ is good, the space $\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1,\theta_n;\gamma,\gamma_q)$ contains a distinguished family of operators of the form $\Phi_\gamma \circ D_u \circ \Phi_\gamma^{-1}$, where $u = u_{\delta,\gamma,-1,\infty}$ is the cylinder corresponding to a semi-infinite gradient trajectory ending at q and $\Phi_\gamma : T\widehat{W}|_{S_\gamma} \to S_\gamma \times \mathbb{R}^{2n}$ is a fixed trivialization satisfying $\Phi_\gamma(X_H) = (1,0) \in \mathbb{R}\oplus\mathbb{R}^{2n-1}$. This family is naturally parametrized by $W^s(q)$, hence it is connected. As seen in Proposition 4.9 the above Fredholm operators are surjective and have index $1-\mathrm{ind}(q)$. If the index is zero we choose the orientation sign to be $+1$. If the index is one the kernel is generated by a nonvanishing section of the form $(f,0)$, hence is canonically isomorphic to $\mathbb{R}\oplus 0$ and therefore admits a canonical orientation. If γ is bad, we choose in an arbitrary way a lift of the operator $\Phi_\gamma \circ D_u \circ \Phi_\gamma^{-1}$, where $u = u_{\delta,\gamma,-1,\infty}$ is the cylinder corresponding to a constant semi-infinite gradient trajectory at q. This determines a lift of the whole path of operators described above, and hence an orientation of $\mathrm{Det}(\widetilde{\widetilde{\mathcal{O}}}^s(\mathbb{R}\times S^1,E;\gamma,\gamma_q))$ by the previous rule.

We induce orientations on $\mathrm{Det}(\mathcal{O}_+(\theta_n))$ by gluing orientations on the line bundles $\mathrm{Det}(\widetilde{\mathcal{O}}_+(\theta_n))$ and $\mathrm{Det}(\widetilde{\mathcal{O}}^s(\theta_n))$. The orientations on $\mathrm{Det}(\widetilde{\mathcal{O}}_+(\theta_n))$ and $\mathrm{Det}(\mathcal{O}_+(\theta_n))$ determine orientations on $\mathrm{Det}(\widetilde{\mathcal{O}}_\pm(E))$ and $\mathrm{Det}(\mathcal{O}_\pm(E))$ by requiring that the glued orientation on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1,E))$ is the canonical one. We get orientations of $\mathrm{Det}(\widetilde{\mathcal{O}}(\mathbb{R}\times S^1,E))$ by requiring that the orientation induced on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1,\theta_n\#E\#\theta_n))$ by the gluing operation

$$\widetilde{\mathcal{O}}_+(\mathbb{C},\theta_n;\overline{\gamma}) \;{}_{\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}} \widetilde{\mathcal{O}}(\mathbb{R}\times S^1,E;\overline{\gamma},\underline{\gamma})_{\underline{\mathrm{ev}}}\times_{\overline{\mathrm{ev}}} \widetilde{\mathcal{O}}_-(\mathbb{C},\theta_n;\underline{\gamma}) \to \mathcal{O}(\mathbb{C}P^1,\theta_n\#E\#\theta_n)$$

is the canonical one. Here we have denoted by $\overline{\mathrm{ev}}$, $\underline{\mathrm{ev}}$ the evaluation maps to S^1 at $-\infty$ and $+\infty$ respectively. Similarly, we get orientations on $\mathrm{Det}(\mathcal{O}(\mathbb{R}\times S^1,E))$, $\mathrm{Det}(\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1,E))$ and $\mathrm{Det}(\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1,E))$, as well as orientations on $\mathrm{Det}(\widetilde{\widetilde{\mathcal{O}}})$ for the various spaces $\widetilde{\widetilde{\mathcal{O}}}$.

Lemma 4.32. *The above recipe defines a system of coherent orientations.*

Proof. We have to prove that, given operators K, L that can be glued lying in one of the spaces $\mathcal{O}$, $\widetilde{\mathcal{O}}$ or $\widetilde{\widetilde{\mathcal{O}}}$, the coherent orientations o_K, o_L of $\mathrm{Det}(K)$ and $\mathrm{Det}(L)$ induce an orientation $o_K\#o_L$ that coincides with the coherent orientation of $\mathrm{Det}(K\#L)$. In the case when K, L belong to some $\mathcal{O}(\mathbb{R}\times S^1)$, $\widetilde{\mathcal{O}}(\mathbb{R}\times S^1)$ or $\widetilde{\widetilde{\mathcal{O}}}(\mathbb{R}\times S^1)$ this means that, for a suitable choice of operators

$A \in \widetilde{\widetilde{\mathcal{O}}}_+(\mathbb{C},\theta_n)$, $A \in \widetilde{\mathcal{O}}_+(\mathbb{C},\theta_n)$ or $A \in \mathcal{O}_+(\mathbb{C},\theta_n)$, and $B \in \widetilde{\widetilde{\mathcal{O}}}_-(\mathbb{C},\theta_n)$, $B \in \widetilde{\mathcal{O}}_-(\mathbb{C},\theta_n)$ or $B \in \mathcal{O}_-(\mathbb{C},\theta_n)$, with o_A, o_B the coherent orientations on the respective determinant line bundles, $o_A\#(o_K\#o_L)\#o_B$ is the canonical orientation on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1))$.

Let E and F be the symplectic vector bundles corresponding to K and L respectively. If $E=\theta_n$ or $F=\theta_n$ the conclusion is a direct consequence of the definitions and of the associativity of gluing. In the general case $E\neq\theta_n$ and $F\neq\theta_n$ we give the proof when $K\in\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1,E;\overline{\gamma}_p,\gamma)$ and $L\in\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1,F;\gamma,\underline{\gamma}_q)$, the other cases being similar. Let us introduce an auxiliary loop of symmetric matrices $S_0\in\mathcal{S}$ such that $[S_0,J_0]=0$, and we define the orientations on $\mathrm{Det}(\mathcal{O}_\pm(\mathbb{C},E';S_0))$ to be the canonical ones (see Remark 4.24). This determines in turn orientations on $\mathrm{Det}(\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1,E';S_0,S_\gamma))$, $\gamma\in\mathcal{P}(H)$ by requiring that gluing induces the coherent orientation on $\mathrm{Det}(\widetilde{\mathcal{O}}_+(\mathbb{C},\theta_n\#E';\gamma))$.

Let $A_1\in\mathcal{O}_+(\mathbb{C},E_1;S_0)$, $K_1\in\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1,\theta_n;S_0,S_\gamma)$ with $E_1\#\theta_n=\theta_n\#E$. By the above definition, we have $o_{A_1}\#o_{K_1}=o_A\#o_K$. We obtain

$$\begin{aligned} o_A\#(o_K\#o_L)\#o_B &= (o_A\#o_K)\#o_L\#o_B=(o_{A_1}\#o_{K_1})\#o_L\#o_B\\ &= o_{A_1}\#o_{K_1\#L}\#o_B=o_{A_1}\#o_{K_1\#L\#B}. \end{aligned}$$

The operators A_1 and $K_1\#L\#B$ are homotopic to $\mathbb{C}$-linear operators with asymptotic condition S_0. The main observation now is that the gluing of two $\mathbb{C}$-linear operators is again $\mathbb{C}$-linear, hence the gluing of the above orientations is the canonical one on $\mathrm{Det}(\mathcal{O}(\mathbb{C}P^1))$. □

Let $\overline{\gamma},\underline{\gamma}$ be good orbits. In this case the procedure for orienting the Morse-Bott spaces of Floer trajectories $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,J)$ is entirely similar to the corresponding procedure in the nondegenerate case (it is actually simpler since we do not need the intermediate transition from $L^{p,d}$ to L^p spaces). Namely, we pull back the orientation on $\mathrm{Det}(\widetilde{\mathcal{O}})$ using the natural map $\widehat{\mathcal{M}}\to\widetilde{\mathcal{O}}$. This in turn induces orientations on the quotient spaces $\mathcal{M}^A(S_{\overline{\gamma}},S_{\underline{\gamma}};H,J)$. Recall that, given oriented vector spaces $V\subset W$, we define an orientation on W/V by requiring that the isomorphism $V\oplus(W/V)\simeq W$ is orientation preserving.

Since the stable and unstable manifolds of the functions f_γ are canonically oriented, one gets orientations (i.e. signs) on all zero-dimensional moduli spaces of Floer trajectories with gradient fragments $\mathcal{M}^A(p,q;H,\{f_\gamma\},J)$ which involve only good orbits. This is done by the following **fibered sum rule**. Let $f_i:W_i\to W$, $i=1,2$ be linear maps of oriented vector spaces such that $f:W_1\oplus W_2\to W$, $(w_1,w_2)\mapsto f_1(w_1)-f_2(w_2)$ is surjective. The orientation on the fibered sum $W_1{}_{f_1}\!\oplus_{f_2}W_2:=\ker f$ is defined such that the isomorphism of vector spaces $(W_1\oplus W_2)/\ker f\xrightarrow{\sim}W$ induced by f changes orientations by the sign $(-1)^{\dim W_2\cdot\dim W}$. Note that this rule is such that the fibered sum operation is associative for oriented vector spaces, and moreover, if f_2 is an orientation preserving isomorphism, the natural isomorphism $W_1{}_{f_1}\!\oplus_{f_2}W_2\simeq W_1$ is orientation preserving. Similarly, if f_1 is an orientation preserving isomorphism, the natural isomorphism $W_1{}_{f_1}\!\oplus_{f_2}W_2\simeq W_2$ is orientation preserving.

The important remark now is that, although the spaces $\widehat{\mathcal{M}}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$ with $\overline{\gamma}$ or $\underline{\gamma}$ being a bad orbit may not be orientable, we can nevertheless define orientations (i.e. signs) on *all* zero-dimensional moduli spaces of Floer trajectories with gradient fragments $\mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$. The sign of an (isolated) point $[\mathbf{u}] = (c_m, [u_m], \ldots, c_1, [u_1], c_0)$ in this moduli space is determined as follows. For each operator D_{u_i}, $i = 1, \ldots, m$ with at least one bad asymptote we choose a lift in the corresponding space $\widetilde{\widetilde{\mathcal{O}}}(\mathbb{R} \times S^1)$. For each c_i, $i = 0, \ldots, m$ lying on a bad orbit γ_i the corresponding operator $D_{u_{\delta, \gamma_i, -T_i/2, T_i/2}}$ admits a unique lift to the space $\widetilde{\widetilde{\mathcal{O}}}(S_{\gamma_i}, S_{\gamma_i})$ such that it can be glued with both $D_{u_{i+1}}$ and D_{u_i}. Since all these operators are surjective, the orientations of the determinant line bundles over the spaces $\widetilde{\mathcal{O}}$ and $\widetilde{\widetilde{\mathcal{O}}}$ induce orientations on $T_{u_i}\widehat{\mathcal{M}}^{A_i}(S_{\gamma_i}, S_{\gamma_{i-1}})$, respectively $TW^u(p)$, $TW^s(q)$ and $T_{(c_i(-T_i/2), T_i)}(S_{\gamma_i} \times \mathbb{R}^+)$, $i = 1, \ldots, m-1$. By the fibered sum rule we get an orientation on $T_{\mathbf{u}}\widehat{\mathcal{M}}^A(p, q; H, \{f_\gamma\}, J)$ which we call "the coherent orientation". On the other hand this vector space carries the "geometric orientation" of the basis $(\partial_s u_m, \ldots, \partial_s u_1)$. We define the sign

$$\epsilon(\mathbf{u}) = \epsilon([\mathbf{u}]) \tag{87}$$

to be $+1$ if these two orientations coincide, and -1 if they are different.

We now want to compare the signs $\epsilon(\mathbf{u})$ with the signs $\epsilon(u_\delta)$ of the glued trajectories u_δ corresponding to $\mathbf{u}$. The situation is expressed by the following diagram, in which we dropped the decorations A, $(H, \{f_\gamma\}, J)$ and (H_δ, J) and in which we have indicated on the morphism arrows the way in which the corresponding isomorphisms of vector spaces act on orientations.

$$\begin{tikzcd}
\begin{array}{c}\text{Coherent}\\ \text{orientation}\end{array} \arrow[r, dotted] & T\widehat{\mathcal{M}}(p,q) \arrow[r, "1", "\phi"'] \arrow[d, "\mathrm{Id}"', "\epsilon(\mathbf{u})"] & T\widehat{\mathcal{M}}(\overline{\gamma}_p, \underline{\gamma}_q) \oplus \mathbb{R}^{m-1} \arrow[d, "\mathrm{Id}"', "\epsilon(u_\delta)"] & \begin{array}{c}\text{Coherent}\\ \text{orientation}\end{array} \arrow[l, dotted] \\
\begin{array}{c}\text{Geometric}\\ \text{orientation}\\ \langle \partial_s u_m, \ldots, \partial_s u_1 \rangle\end{array} \arrow[r, dotted] & T\widehat{\mathcal{M}}(p,q) \arrow[r, "?", "\phi"'] & T\widehat{\mathcal{M}}(\overline{\gamma}_p, \underline{\gamma}_q) \oplus \mathbb{R}^{m-1} & \begin{array}{c}\text{Geometric}\\ \text{orientation}\\ \langle \partial_s u_\delta \rangle \oplus \mathbb{R}^{m-1}\end{array} \arrow[l, dotted]
\end{tikzcd}$$

The map ϕ is defined from gluing as follows. The tangent space $T\widehat{\mathcal{M}}(p, q)$ is the kernel of the operator $D_{\widetilde{w}}$, $\widetilde{w} = (v_m, u_m, \ldots, v_1, u_1, v_0)$ considered in Lemma 4.17. Moreover, since the cokernel of $D_{\widetilde{w}}$ is naturally oriented, the coherent orientation of $\mathrm{Det}(D_{\widetilde{w}})$ induces a "coherent" orientation on $\ker D_{\widetilde{w}}$. Recall that the analytical expression of $D_{\widetilde{w}}$ is $D_{v_m} \oplus D_{u_m} \oplus D'_{v_{m-1}} \oplus \ldots \oplus D'_{v_1} \oplus D_{u_1} \oplus D_{v_0}$, and note that $D_{\widetilde{w}}$ admits a natural stabilization $D_{\widetilde{w}}^{\mathbb{R}^{m-1}}$ obtained by replacing D'_{v_i}, $i = 1, \ldots, m-1$ with $D\{\bar{\partial}_T\}(v_i, T_{v_i})$ (see Remark 4.11 for the definitions). By [5, Corollary 6] there is a natural isomorphism $\widetilde{\phi} : \ker D_{\widetilde{w}}^{\mathbb{R}^{m-1}} \xrightarrow{\sim} \ker D_{G_\delta(\widetilde{w})}^{\mathbb{R}^{m-1}}$ which preserves the coherent orientations. We denote by $\phi : \ker D_{\widetilde{w}}^{\mathbb{R}^{m-1}} \xrightarrow{\sim} \ker D_{u_\delta}^{\mathbb{R}^{m-1}}$ the composition of $\widetilde{\phi}$ with the projection Π on $\ker D_{u_\delta}^{\mathbb{R}^{m-1}}$ along the image of the right inverse Q_δ of $D_{G_\delta(\widetilde{w})}$ given by Proposition 4.18. Since $D_{G_\delta(\widetilde{w})}$

and D_{u_δ} are close in the relevant δ-norm, we get that ϕ is an isomorphism preserving coherent orientations.

The vertical maps change orientations by $\epsilon(\mathbf{u})$, respectively $\epsilon(u_\delta)$ by definition, and the whole work now goes into determining the action of ϕ on the geometric orientations.

Remark 4.33. If γ is a good orbit and $p \in \mathrm{Crit}(f_\gamma)$, the geometric orientations on $W^u(p)$ and $W^s(p)$ coincide with the coherent ones. Indeed, the unstable manifold $W^u(p)$ is naturally identified with the zero set of the section $\bar\partial_{-\infty,1}$ defined on $\mathcal{B}^{1,p,d}_\delta(p, S_\gamma; f_\gamma)$ by (54), whereas the stable manifold $W^s(p)$ is naturally identified with the zero set of the section $\bar\partial_{-1,\infty}$ defined on $\mathcal{B}^{1,p,d}_\delta(S_\gamma, q; f_\gamma)$. The assertion for $W^s(p)$ is then a direct consequence of the definition of the orientation on $\mathrm{Det}(\widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, \theta_n; \gamma, \gamma_q))$. As for $W^u(p)$, let us consider the gluing operation

$$\widetilde{\mathcal{O}}^u(\mathbb{R}\times S^1, \theta_n; \gamma_p, \gamma)_{\underline{\mathrm{ev}}} \times_{\overline{\mathrm{ev}}} \widetilde{\mathcal{O}}^s(\mathbb{R}\times S^1, \theta_n; \gamma, \gamma_p) \to \mathcal{O}(\mathbb{R}\times S^1, \theta_n; \gamma_p, \gamma_p).$$

We choose the surjective operators $D_1 := D_{u_{\delta,\gamma,-\infty,1}}$, $D_2 := D_{u_{\delta,\gamma,-1,\infty}}$ corresponding to the constant gradient trajectory at p. With these choices $D_1 \# D_2 = D_{u_{\delta,\gamma,-\infty,\infty}} =: D$ also corresponds to the constant gradient trajectory at p. The operator D is an isomorphism and, by the coherent choice of the orientations, the determinant line $\mathrm{Det}(D) \simeq \mathbb{R}$ is positively oriented. If p is the maximum of f_γ then $\ker D_2 \simeq T_p S_\gamma$ as oriented vector spaces (by definition), the kernel of D_1 is trivial and its determinant line must be positively oriented. If p is the minimum of f_γ then $\ker D_2$ is trivial and its determinant line is positively oriented by definition, therefore $\ker D_1 \simeq T_p S_\gamma$ must have the geometric orientation.

Lemma 4.34. *Assume* $\dim \mathcal{M}^A(p, q; H, \{f_\gamma\}, J) = 0$ *and fix an element* $[\mathbf{u}] \in \mathcal{M}^A(p, q; H, \{f_\gamma\}, J)$ *with* $m \le 2$ *sublevels. Then* $\epsilon(\mathbf{u}) = \epsilon(u_\delta)$ *if* $m = 0, 1$ *and* $\epsilon(\mathbf{u}) = -\epsilon(u_\delta)$ *if* $m = 2$.

Proof. If $m = 0$ the statement is obvious since $\mathbf{u}$ consists of a single gradient trajectory and $\mathbf{u} = u_\delta$ (see Lemma 4.28). We now have to show that the map ϕ in the previous diagram preserves the geometric orientation if $m = 1$, respectively reverses it if $m = 2$. Since a shift σ on u_1 produces a glued trajectory u_δ shifted by the same amount σ, we infer that $\phi(\partial_s u_1) = \partial_s u_\delta$ and, for $m = 1$, the statement follows from the commutativity of the diagram.

Let us now examine the case $m = 2$. We recall that $\phi = \Pi \circ \widetilde{\phi}$, where the isomorphism $\widetilde{\phi}$ is the composition of the gluing map G in the proof of Proposition 4.18 with the projection to $\ker D^{\mathbb{R}}_{G_\delta(\widetilde{w})}$ along the image of Q_δ (see [5]). We first show that $\phi(\partial_s u_1 + \partial_s u_2)$ is close in $\|\cdot\|_{1,\delta}$-norm to $\partial_s u_\delta$. We denote $\widetilde{w}^{\sigma,\sigma} := (v_2, u_2(\cdot+\sigma), v_1, u_1(\cdot+\sigma), v_0)$, $\sigma \in \mathbb{R}$. Then $G(0 \oplus \partial_s u_2 \oplus 0 \oplus \partial_s u_1 \oplus 0)$ is $\|\cdot\|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w}^{\sigma,\sigma})$, which is $\|\cdot\|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w})(\cdot+\sigma)$, which is in turn close to $\frac{d}{d\sigma}\big|_{\sigma=0} v_\delta(\cdot+\sigma) = \partial_s v_\delta$. Then $\phi(\partial_s u_1 + \partial_s u_2) = \Pi(G(0 \oplus \partial_s u_2 \oplus 0 \oplus \partial_s u_1 \oplus 0))$ is $\|\cdot\|_{1,\delta}$-close to $\Pi(\partial_s u_\delta) = \partial_s u_\delta$.

We now show that $\phi(\partial_s u_1 - \partial_s u_2) \in \ker D_{u_\delta} \oplus \mathbb{R}$ is a vector having a negative component in the $\mathbb{R}$ direction and whose component on $\ker D_{u_\delta}$ is $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_1$. Then the conclusion follows.

Let $\widetilde{w}^{-\sigma,\sigma} := (v_2, u_2(\cdot - \sigma), v_1, u_1(\cdot + \sigma), v_0)$, $\sigma \in \mathbb{R}$. Then $G(0 \oplus (-\partial_s u_2) \oplus 0 \oplus \partial_s u_1 \oplus 0)$ is $\|\cdot\|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w}^{-\sigma,\sigma})$. We define $\epsilon(\sigma) := 2\delta\sigma$ and the section

$$\frac{d}{d\sigma}\Big|_{\sigma=0} G_{\delta,\epsilon(\sigma)}(\widetilde{w}^{-\sigma,\sigma}) = \frac{d}{d\sigma}\Big|_{\sigma=0} G_\delta(\widetilde{w}^{-\sigma,\sigma}) + 2\delta \frac{d}{d\epsilon}\Big|_{\epsilon=0} G_{\delta,\epsilon}(\widetilde{w}) \tag{88}$$

is by construction $\|\cdot\|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w}^{-\sigma,-\sigma})$, hence $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_\delta$. By adapting the arguments in the proof of Proposition 4.15 one sees that the section

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \bar\partial_{T_{v_1}+\epsilon} G_{\delta,\epsilon}(\widetilde{w}) = D\bar\partial_{T_{v_1}} \frac{d}{d\epsilon}\Big|_{\epsilon=0} G_{\delta,\epsilon}(\widetilde{w}) + D\{\bar\partial_T\}(G_\delta(\widetilde{w}), T_{v_1}) \cdot (0,1)$$

is $\|\cdot\|_\delta$-small. Here the sections $\bar\partial_T$ are of the form $\bar\partial_{H_T,J}$, where H_T is the s-dependent Hamiltonian given respectively by (54) on the intervals of definition of v_2, v_1, v_0, and equal to H on the intervals of definition of u_1, u_2. The previous equation shows that $(\frac{d}{d\epsilon}\big|_{\epsilon=0} G_{\delta,\epsilon}(\widetilde{w}), 1) \in \mathrm{dom}(D^{\mathbb{R}}_{G_\delta(\widetilde{w})})$ is $\|\cdot\|_{1,\delta}$-close to $\ker D^{\mathbb{R}}_{u_\delta}$. On the other hand, equation (88) shows that $G(0 \oplus (-\partial_s u_2) \oplus 0 \oplus \partial_s u_1 \oplus 0)$ is $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_\delta - 2\delta \frac{d}{d\epsilon}\big|_{\epsilon=0} G_{\delta,\epsilon}(\widetilde{w})$. Hence, after projecting to $\ker D^{\mathbb{R}}_{u_\delta} = \ker D_{u_\delta} \oplus \mathbb{R}$, we get a vector having a negative component in the $\mathbb{R}$ direction and whose component on $\ker D_{u_\delta}$ is $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_\delta$. □

Proof of Proposition 3.9. The special statement concerning the case $m = 0$ was proved in Lemmas 4.28 and 4.34, whereas the equality $\epsilon(\mathbf{u}) = (-1)^{m-1}\epsilon(u_\delta)$ in case $m = 1, 2$ was the content of Lemma 4.34. The proof in the case $m \geq 3$ is just a more elaborate version of the proof of Lemma 4.34. We consider the basis of $T_{\mathbf{u}}\widehat{\mathcal{M}}(p,q)$ given by

$$\begin{aligned}
e_0 &:= \partial_s u_m + \partial_s u_{m-1} + \ldots + \partial_s u_2 + \partial_s u_1, \\
e_1 &:= -\partial_s u_m + \partial_s u_{m-1} + \ldots + \partial_s u_2 + \partial_s u_1, \\
&\ \ \vdots \\
e_{m-2} &:= -\partial_s u_m - \partial_s u_{m-1} - \ldots + \partial_s u_2 + \partial_s u_1, \\
e_{m-1} &:= -\partial_s u_m - \partial_s u_{m-1} - \ldots - \partial_s u_2 + \partial_s u_1.
\end{aligned}$$

It is easy to see that the orientation determined by $(e_0, \ldots, e_{m-1})$ is the same as the geometric orientation determined by $(\partial_s u_m, \ldots, \partial_s u_1)$. We have to show that the orientation of the basis $(\phi(e_0), \ldots, \phi(e_{m-1}))$ differs from the canonical orientation of $\langle \partial_s u_\delta \rangle \oplus \mathbb{R}^{m-1}$ by $(-1)^{m-1}$.

As in Lemma 4.34 we see that $\phi(e_0)$ is $\|\cdot\|_{1,\delta}$-close to $\partial_s u_\delta$. We now show that $\phi(e_k) \in \ker D_{u_\delta} \oplus \mathbb{R}^{m-1}$, $k = 1, \ldots, m-1$ has a negative component which is bounded away from zero along the corresponding factor $\mathbb{R} \subset \mathbb{R}^{m-1}$, that the other components in $\mathbb{R}^{m-1}$ are close to zero, whereas the component along $\ker D_{u_\delta}$ is close to $-\partial_s u_\delta$ in $\|\cdot\|_{1,\delta}$-norm. Then the conclusion will follow since the orientation defined by $(\phi(e_0), \ldots, \phi(e_{m-1}))$ is the same as the orientation defined by

$$(\partial_s u_\delta, 0, \ldots, 0), (-\partial_s u_\delta, -1, 0, \ldots, 0), \ldots, (-\partial_s u_\delta, 0, \ldots, 0, -1)).$$

Let us fix $k = 1, \dots, m-1$. We shall freely use the notation e_k for the vector $0 \oplus (-\partial_s u_m) \oplus 0 \oplus \dots \oplus (-\partial_s u_{m-k+1}) \oplus 0 \oplus \partial_s u_{m-k} \oplus \dots \partial_s u_1 \oplus 0$ in the domain of the gluing map G defined in the proof of Proposition 4.18. For $\sigma > 0$ we denote

$$\widetilde{w}_k^{-\sigma,\sigma} := (v_m, u_m(\cdot - \sigma), \dots, u_{m-k+1}(\cdot - \sigma), v_{m-k}, u_{m-k}(\cdot + \sigma), \dots, u_1(\cdot + \sigma), v_0),$$

$$\widetilde{w}^{-\sigma,-\sigma} := (v_m, u_m(\cdot - \sigma), \dots, u_{m-k+1}(\cdot - \sigma), v_{m-k}, u_{m-k}(\cdot - \sigma), \dots, u_1(\cdot - \sigma), v_0).$$

Then $G(e_k)$ is $\| \cdot \|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w}_k^{-\sigma,\sigma})$. We denote

$$\overline{\epsilon}_k(\epsilon) := (0, \dots, \epsilon, \dots, 0),$$

where the parameter $\epsilon > 0$ appears on position $m-k$. The section

$$\frac{d}{d\sigma}\Big|_{\sigma=0} G_{\delta, \overline{\epsilon}_k(2\delta\sigma)}(\widetilde{w}_k^{-\sigma,\sigma}) = \frac{d}{d\sigma}\Big|_{\sigma=0} G_\delta(\widetilde{w}_k^{-\sigma,\sigma}) + 2\delta \frac{d}{d\epsilon}\Big|_{\epsilon=0} G_{\delta,\overline{\epsilon}_k}(\widetilde{w}) \tag{89}$$

is by construction $\|\cdot\|_{1,\delta}$-close to $\frac{d}{d\sigma}\big|_{\sigma=0} G_\delta(\widetilde{w}^{-\sigma,-\sigma})$, hence $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_\delta$. As in Lemma 4.34, by adapting the arguments in the proof of Proposition 4.15 one sees that the section

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \bar\partial_{T_{v_{m-k}}+\epsilon} G_{\delta,\overline{\epsilon}_k(\epsilon)}(\widetilde{w}) \quad &= \quad D\bar\partial_{T_{v_{m-k}}} \frac{d}{d\epsilon}\Big|_{\epsilon=0} G_{\delta,\overline{\epsilon}_k(\epsilon)}(\widetilde{w}) \\ &\quad + D\{\bar\partial_T\}(G_\delta(\widetilde{w}), T_{v_{m-k}}) \cdot (0,1) \end{aligned}$$

is $\| \cdot \|_\delta$-small. As before, the sections $\bar\partial_T$ are of the form $\bar\partial_{H_T,J}$, where H_T is the s-dependent Hamiltonian given respectively by (54) on the intervals of definition of $v_m, v_{m-1}, \dots, v_0$, and equal to H on the intervals of definition of $u_m, \dots, u_1$. The previous equation shows that

$$(\frac{d}{d\epsilon}\Big|_{\epsilon=0} G_{\delta,\overline{\epsilon}_k(\epsilon)}(\widetilde{w}), 0, \dots, 1, \dots, 0) \in \mathrm{dom}(D^{\mathbb{R}^{m-1}}_{G_\delta(\widetilde{w})})$$

is $\|\cdot\|_{1,\delta}$-close to $\ker D^{\mathbb{R}^{m-1}}_{u_\delta}$. On the other hand, equation (89) shows that $G(e_k)$ is $\|\cdot\|_{1,\delta}$-close to $-\partial_s u_\delta - 2\delta \frac{d}{d\epsilon}\big|_{\epsilon=0} G_{\delta,\overline{\epsilon}_k(\epsilon)}(\widetilde{w})$. After projecting to $\ker D^{\mathbb{R}^{m-1}}_{u_\delta}$ we get a vector whose k-th component in $\mathbb{R}^{m-1}$ is negative, whose other components in $\mathbb{R}^{m-1}$ are small, and whose component on $\ker D_{u_\delta}$ is $\| \cdot \|_{1,\delta}$-close to $-\partial_s u_\delta$. □

Remark 4.35. We chose to define the signs $\epsilon(\mathbf{u})$ by comparing the orientation induced on $T_{\mathbf{u}}\widehat{\mathcal{M}}^A(p,q;H,\{f_\gamma\},J)$ by the fiber sum rule from the coherent orientations on $T\widehat{\mathcal{M}}^{A_i}(S_{\gamma_i}, S_{\gamma_{i-1}};H,J)$, $i = 1, \dots, m$ with the orientation of the basis $(\partial_s u_m, \dots, \partial_s u_1)$. Another possible recipe would have been the following: induce orientations on $T\mathcal{M}^{A_i}(S_{\gamma_i}, S_{\gamma_{i-1}};H,J)$ out of the coherent orientations by quotienting out $\langle \partial_s \rangle$, then apply the fiber sum rule in order to get a sign on the zero-dimensional spaces $T_{[\mathbf{u}]}\mathcal{M}^A(p,q;H,\{f_\gamma\},J)$. The sign obtained in this way would have differed from the previously defined $\epsilon(\mathbf{u})$ by a factor ± 1 which

can be explicitly computed and which depends on the combinatorics of the levels of $\mathbf{u}$. The curious reader can test this procedure in the case $m = 1$: it gives a sign equal to $\epsilon(\mathbf{u})$ if p, q are both minima, respectively equal to $-\epsilon(\mathbf{u})$ if p, q are both maxima. The following two properties of the fibered sum constitute a useful tool for making the verification (here W_1 and W_2 are oriented vector spaces).

- the natural isomorphism $W_1 \,{}_0\!\bigoplus_{f_2} W_2 \xrightarrow{\sim} W_1 \oplus \ker f_2$ changes the orientation by $(-1)^{\dim W_1 \cdot (\dim W_2 + 1)}$;
- the natural isomorphism $W_1 \,{}_{f_1}\!\bigoplus_0 W_2 \xrightarrow{\sim} \ker f_1 \oplus W_2$ preserves the orientation.

A Appendix: Asymptotic estimates

For all $\gamma \in \mathcal{P}(H)$, we choose coordinates $(\vartheta, z) \in S^1 \times \mathbb{R}^{2n-1}$ parametrizing a tubular neighbourhood of γ, such that $\vartheta \circ \gamma(\theta) = \theta$ and $z \circ \gamma(\theta) = 0$. Given a smooth function $f_\gamma : S_\gamma \to \mathbb{R}$, we denote by $\varphi_s^{f_\gamma}$ the gradient flow of f_γ with respect to the natural metric on S^1.

In a neighbourhood of $\gamma \in \mathcal{P}(H)$ the Floer equation $\partial_s u + J\partial_\theta u - JX_H = 0$ becomes $\partial_s Z + J\partial_\theta Z + J\frac{\partial}{\partial\vartheta} - JX_H = 0$, where $Z(s,\theta) := (\vartheta \circ u(s,\theta) - \theta, z \circ u(s,\theta))$. Since $X_H = \frac{\partial}{\partial\vartheta}$ on $\{z = 0\}$ this can be rewritten as

$$\partial_s Z + J\partial_\theta Z + Sz = 0$$

for some matrix-valued function $S = S(\vartheta, z)$. The matrix $S_\infty(\theta) := S(\theta, 0)$ is symmetric. Let $A_\infty : H^k(S^1, \mathbb{R}^{2n}) \to H^{k-1}(S^1, \mathbb{R}^{2n})$ be the operator defined by

$$A_\infty Z := J\frac{d}{d\theta} Z + S_\infty(\theta) z.$$

The kernel of A_∞ has dimension one and is spanned by the constant vector $e_1 := (1, 0, \ldots, 0)$. We denote by Q_∞ the orthogonal projection onto $(\ker A_\infty)^\perp$ and we set $P_\infty := \mathbb{1} - Q_\infty$. Then A_∞ is invertible when restricted to $\operatorname{im} Q_\infty$ and $Q_\infty A_\infty = A_\infty$.

Proposition A.1. *Let $H \in \mathcal{H}'$ be fixed. There exists $r > 0$ such that for all $J \in \mathcal{J}^\ell$ and for all $u \in \mathcal{M}^A(S_{\overline{\gamma}}, S_{\underline{\gamma}}; H, J)$, $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}(H)$ we have*

$$\begin{aligned}
\vartheta \circ u(s,\theta) - \theta - \overline{\theta}_0 &\in W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}; e^{r|s|} ds\, d\theta), \\
z \circ u(s,\theta) &\in W^{1,p}(]-\infty, -s_0] \times S^1, \mathbb{R}^{2n-1}; e^{r|s|} ds\, d\theta), \\
\vartheta \circ u(s,\theta) - \theta - \underline{\theta}_0 &\in W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}; e^{r|s|} ds\, d\theta), \\
z \circ u(s,\theta) &\in W^{1,p}([s_0, \infty[\times S^1, \mathbb{R}^{2n-1}; e^{r|s|} ds\, d\theta),
\end{aligned}$$

for some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ and some $s_0 > 0$ sufficiently large.

Proof. We make the proof only at $+\infty$ since the case of $-\infty$ is entirely similar. For s large enough we set $S(s,\theta) := S(\vartheta \circ u(s,\theta), z \circ u(s,\theta))$, so that $S_\infty(\theta) = \lim_{s\to\infty} S(s,\theta)$ and $\lim_{s\to\infty} |\partial_s S(s,\theta)| = 0$.

Let $A(s) : H^k(S^1, \mathbb{R}^{2n}) \to H^{k-1}(S^1, \mathbb{R}^{2n})$ be the operator defined by

$$A(s)Z := J\frac{d}{d\theta}Z + S(s,\theta)z,$$

so that $A_\infty = \lim_{s\to\infty} A(s)$. We have $A(s) = A(s)Q_\infty$, $\partial_s Q_\infty = Q_\infty \partial_s$. Since A_∞ is invertible when restricted to $\operatorname{im} Q_\infty$ and $Q_\infty A_\infty = A_\infty$, the operators $A(s)$ and $Q_\infty A(s)$ are also invertible when restricted to $\operatorname{im} Q_\infty$ for s large enough and there exists $c > 0$ such that

$$\|A(s)Q_\infty Z\|^2 \geq \|Q_\infty A(s) Q_\infty Z\|^2 \geq c\|Q_\infty Z\|^2$$

for all $Z \in H^k(S^1, \mathbb{R}^{2n})$. For s large enough we define

$$f(s) := \frac{1}{2}\|Q_\infty Z(s)\|^2.$$

We have

$$\begin{aligned}
f''(s) &= \|\partial_s Q_\infty Z\|^2 + \langle Q_\infty Z, \partial_s^2 Q_\infty Z\rangle \\
&= \|\partial_s Q_\infty Z\|^2 - \langle Q_\infty Z, \partial_s Q_\infty A(s) Q_\infty Z\rangle \\
&= \|Q_\infty A(s) Q_\infty Z\|^2 - \langle Q_\infty Z, Q_\infty(\partial_s A(s)) Q_\infty Z - Q_\infty A(s)^2 Q_\infty Z\rangle \\
&\geq (c-\varepsilon)\|Q_\infty Z\|^2 + \langle (A(s)^* - A(s))Q_\infty Z, A(s)Q_\infty Z\rangle + \|A(s)Q_\infty Z\|^2 \\
&\geq (2c - 2\varepsilon)\|Q_\infty Z\|^2 \geq 4\rho^2 f(s).
\end{aligned}$$

Here $A(s)^*$ is the adjoint of $A(s)$ and we used the fact that $\|\partial_s A(s)\| \to 0$, $A(s)^* - A(s) \to 0$ for $s \to \infty$ and $\|A(s)\|$ is uniformly bounded.

Let now s_0 be large enough and define $g(s) := f(s_0)e^{-2\rho(s-s_0)}$. Then $g'' = 4\rho^2 g$, $(f-g)'' \geq 4\rho^2(f-g)$, $(f-g)(s_0) = 0$ and $\lim_{s\to\infty} f(s) - g(s) = 0$. Then $f - g \leq 0$ on $[s_0, \infty[$ because it cannot have a strictly positive maximum. Therefore

$$\|Q_\infty Z(s)\| \leq \|Q_\infty Z(s_0)\| e^{-\rho(s-s_0)}.$$

It is important to note that this estimate holds for any Sobolev norm H^k. By the Sobolev embedding theorem this implies the following pointwise estimate

$$|Q_\infty Z(s,\theta)| \leq Ce^{-\rho s}, \quad |\partial_\theta Q_\infty Z(s,\theta)| = |\partial_\theta Z(s,\theta)| \leq Ce^{-\rho s}, \quad s \geq s_0.$$

Because $\partial_s Z + A(s)Z = \partial_s Z + A(s)Q_\infty Z = 0$ we obtain

$$|\partial_s Z(s,\theta)| \leq Ce^{-\rho s}, \quad s \geq s_0$$

and, by integration on $[s, \infty[$ and taking into account that $Z(s,\theta)$ converges to $(\underline{\theta}_0, 0, \ldots, 0)$ for $s \to \infty$, we obtain the pointwise estimate

$$|(\vartheta - \theta - \underline{\theta}_0, z)| \leq Ce^{-\rho s}.$$

This implies the conclusion for any $r < \rho$. $\square$

Proposition A.2. *Let $H \in \mathcal{H}'$ and $\{f_\gamma : S_\gamma \to \mathbb{R}\}$ be a collection of perfect Morse functions indexed by $\gamma \in \mathcal{P}_\lambda$. There exist $r > 0$ and $\delta_0 > 0$ such that for all $J \in \mathcal{J}$, $\overline{\gamma}, \underline{\gamma} \in \mathcal{P}(H)$, $p \in \mathrm{Crit}(f_{\overline{\gamma}}), q \in \mathrm{Crit}(f_{\underline{\gamma}})$ and for all $(\delta, u) \in \mathcal{M}^A_{]0,\delta_0]}(\overline{\gamma}_p, \underline{\gamma}_q; H, \{f_\gamma\}, J)$, we have*

$$\begin{array}{rcl} \vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_{\overline{\gamma}}}(\overline{\theta}_0) & \in & W^{1,p}(]-\infty,-s_0] \times S^1, \mathbb{R}; e^{r|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}(]-\infty,-s_0] \times S^1, \mathbb{R}^{2n-1}; e^{r|s|} ds\, d\theta), \\ \vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_{\underline{\gamma}}}(\underline{\theta}_0) & \in & W^{1,p}([s_0,\infty[\times S^1, \mathbb{R}; e^{r|s|} ds\, d\theta), \\ z \circ u(s,\theta) & \in & W^{1,p}([s_0,\infty[\times S^1, \mathbb{R}^{2n-1}; e^{r|s|} ds\, d\theta), \end{array}$$

for some $\overline{\theta}_0, \underline{\theta}_0 \in S^1$ and some $s_0 > 0$ sufficiently large.

Proof. The proof is similar to the one of Proposition A.1. With the same notations as before the Floer equation satisfied by u can be written in local coordinates $Z = (\vartheta - \theta, z)$ as

$$\partial_s Z + J\partial_\theta Z + Sz - \delta \nabla f_{\underline{\gamma}}(Z_1) = 0, \tag{90}$$

where $Z_1 := \vartheta - \theta$. We again show that $f(s) = \frac{1}{2}\|Q_\infty Z\|^2$ satisfies an inequality of the form $f''(s) \geq 4\rho^2 f(s)$. There are two additional terms to estimate in the expression of $f''(s)$, namely

$$\langle Q_\infty Z, \delta Q_\infty A(s) \nabla f_{\underline{\gamma}}(Z_1)\rangle \tag{91}$$

and

$$\langle Q_\infty Z, \delta Q_\infty \partial_s (\nabla f_{\underline{\gamma}}(Z_1))\rangle. \tag{92}$$

Let $P_\infty := \mathbb{1} - Q_\infty$ be the orthogonal projection on $\ker A_\infty$. The main observation is that $Q_\infty \nabla f_{\underline{\gamma}}(P_\infty Z_1) = 0$. As a consequence there exists a matrix-valued function $L = L(s,\theta)$ such that

$$Q_\infty \nabla f_{\underline{\gamma}}(Z_1) = L Q_\infty (Z_1).$$

The term (91) is then estimated by

$$\begin{array}{rcl} \langle Q_\infty Z, \delta Q_\infty A(s) \nabla f_{\underline{\gamma}}(Z_1)\rangle & = & \langle Q_\infty Z, \delta Q_\infty A(s) Q_\infty \nabla f_{\underline{\gamma}}(Z_1)\rangle \\ & \leq & C\delta \|Q_\infty Z\|^2 \end{array}$$

for $s \geq s_0$, where s_0 depends on u, but C depends only on $\underline{\gamma}$ and $f_{\underline{\gamma}}$. Similarly, the term (92) is estimated by

$$\begin{array}{rcl} \langle Q_\infty Z, \delta Q_\infty \partial_s (\nabla f_{\underline{\gamma}}(Z_1))\rangle & = & \langle Q_\infty Z, \delta \partial_s Q_\infty \nabla f_{\underline{\gamma}}(Z_1)\rangle \\ & \leq & C\delta \|Q_\infty Z\| \|\partial_s Q_\infty (Z_1)\| \\ & \leq & \dfrac{C\delta}{2}\Big(\|Q_\infty Z\|^2 + \|\partial_s Q_\infty Z\|^2\Big). \end{array}$$

The norm of $\partial_s Q_\infty Z = Q_\infty \partial_s Z$ satisfies

$$\|\partial_s Q_\infty Z\| = \|Q_\infty A(s) Z - \delta Q_\infty \nabla f_{\underline{\gamma}}(Z_1)\| \leq C\|Q_\infty Z\|.$$

As a consequence, there exists $\delta_0 > 0$ and $\rho > 0$ such that $f''(s) \geq 4\rho^2 f(s)$ for $s \geq s_0$ and $0 < \delta \leq \delta_0$. As before, we infer the pointwise bounds

$$|Q_\infty Z(s,\theta)| \leq Ce^{-\rho s},\ |\partial_\theta Q_\infty Z(s,\theta)| = |\partial_\theta Z(s,\theta)| \leq Ce^{-\rho s},\ s \geq s_0. \tag{93}$$

It remains to estimate $P_\infty Z$. For that we write $\nabla f_{\underline{\gamma}}(Z_1) = \nabla f_{\underline{\gamma}}(P_\infty(Z_1)) + KQ_\infty(Z_1)$ for some matrix-valued function $K = K(s,\theta)$. Again, for $s \geq s_0$, the norm $\|K\|$ is uniformly bounded by a constant depending only on $\underline{\gamma}$ and $f_{\underline{\gamma}}$. By applying P_∞ to the equation (90) and using the fact that $P_\infty \nabla f_{\underline{\gamma}}(P_\infty(Z_1)) = \nabla f_{\underline{\gamma}}(P_\infty(Z_1))$ and $P_\infty(Z_1) = P_\infty(Z)$ we obtain

$$|\partial_s(P_\infty Z) - \delta \nabla f_{\underline{\gamma}}(P_\infty Z)| \leq Ce^{-\rho s}. \tag{94}$$

We claim that this implies

$$|P_\infty Z(s) - \varphi_s^{\delta f_{\underline{\gamma}}}(\underline{\theta}_0)| \leq Ce^{-\rho s},\ s \geq s_0 \tag{95}$$

for a suitable $\underline{\theta}_0$. We choose a Morse coordinate x on $S_{\underline{\gamma}}$ around the critical point q of $f_{\underline{\gamma}}$ in which the gradient $\nabla f_{\underline{\gamma}}(x) = \pm Mx$, $M > 0$. Then equation (94) becomes

$$\partial_s(P_\infty Z)(s) \mp \delta M P_\infty Z(s) = G(s)$$

with $|G(s)| \leq Ce^{-\rho s}$. Then $P_\infty Z(s) = c(s)e^{\pm\delta M s}$ with $e^{\pm \delta M s}\partial_s c(s) = G(s)$. As a consequence, for $\delta < \rho/M$ the function c admits a limit c_∞ as $s \to \infty$ and $c(s) = c_\infty - \int_s^\infty G(\sigma)e^{\mp\delta M\sigma}\, d\sigma$. Let $\underline{\theta}_0$ be such that $\varphi_s^{\delta f_{\underline{\gamma}}}(\underline{\theta}_0) = c_\infty e^{\pm\delta M s}$ (note that $c_\infty = 0$ if q is a maximum). Then

$$\begin{aligned} |P_\infty Z(s) - \varphi_s^{\delta f_{\underline{\gamma}}}(\underline{\theta}_0)| &= |e^{\pm\delta M s}\int_s^\infty G(\sigma)e^{\mp\delta M\sigma}\, d\sigma| \\ &\leq Ce^{-\rho s}. \end{aligned}$$

The estimates (93) and (95) imply the conclusion. □

Proposition A.3. *Let $\delta \in]0,\delta_0]$ and let $u_\delta \in \widehat{\mathcal{M}}^A(\overline{\gamma}_p, \underline{\gamma}_q; H_\delta, J)$. Let $I_\delta = [s_0(\delta), s_1(\delta)] \subset \mathbb{R}$ be an interval such that $u_\delta(I_\delta \times S^1)$ is contained in the domain of a coordinate chart $Z = (\vartheta, z)$ around S_γ for some $\gamma \in \mathcal{P}(H)$.*

There exist $\rho > 0$, $\theta_0 \in S^1$, $C > 0$ and $M > 0$ such that $z \circ u(s,\theta)$ and its (first order) derivatives are bounded by

$$C\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\frac{\cosh(\rho(s - \frac{s_0+s_1}{2}))}{\cosh(\rho(s_1 - s_0)/2)} \tag{96}$$

for $s \in I_\delta$, $\theta \in S^1$. If $P_\infty Z(s)$, $s \in I_\delta$ stays away from all but one of the critical points of f_γ, then $\vartheta \circ u(s,\theta) - \theta - \varphi_s^{\delta f_\gamma}(\theta_0)$ and its (first order) derivatives are bounded by

$$C\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)e^{\delta M(s_1 - s_0)}\frac{\cosh(\rho(s - \frac{s_0+s_1}{2}))}{\cosh(\rho(s_1 - s_0)/2)}.$$

Moreover, if $P_\infty Z(s)$, $s \in I_\delta$ stays away from all critical points of f_γ, the above bound is improved to (96).

Proof. With the notations of Proposition A.2, the Floer equation satisfied by u can be written in local coordinates $Z = (\vartheta - \theta, z)$ as

$$\partial_s Z + J\partial_\theta Z + Sz - \delta\nabla f_\gamma(Z_1) = 0, \tag{97}$$

where $Z_1 := \vartheta - \theta$. Let $A_\infty = J\frac{d}{d\theta} + S_\infty(\theta)$ the asymptotic operator at γ, let Q_∞ be the orthogonal projection onto $(\ker A_\infty)^\perp$ and $P_\infty := \mathbb{1} - Q_\infty$. Then, as in Proposition A.2, the quantity $f(s) = \frac{1}{2}\|Q_\infty Z\|^2$ satisfies an inequality of the form $f''(s) \geq 4\rho^2 f(s)$. Define

$$g(s) := \max(f(s_0), f(s_1))\frac{\cosh(2\rho(s - \frac{s_0+s_1}{2}))}{\cosh(\rho(s_1 - s_0))}.$$

Then $(f-g)'' \geq 4\rho^2(f-g)$ and $f-g$ cannot have a strictly positive maximum. Since $f-g \leq 0$ at s_0 and s_1, we infer that $f-g \leq 0$ on I_δ. As in Proposition A.1, we infer the pointwise bounds for $s \geq s_0$

$$\begin{aligned} |Q_\infty Z(s,\theta)| &\leq Cg_1(s), \\ |\partial_\theta Q_\infty Z(s,\theta)| = |\partial_\theta Z(s,\theta)| &\leq Cg_1(s), \\ |\partial_s(P_\infty Z)(s) - \delta\nabla f_\gamma(P_\infty Z)(s)| &\leq C_1 g_1(s), \end{aligned} \tag{98}$$

where

$$g_1(s) := \max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\sqrt{\frac{\cosh(2\rho(s - \frac{s_0+s_1}{2}))}{\cosh(\rho(s_1 - s_0))}}.$$

If $P_\infty Z(s)$ stays away from $\mathrm{Crit}(f_\gamma)$, we can assume that $\nabla f_\gamma(P_\infty Z(s)) = M$ in some suitable coordinate on S^1. Then the last equation becomes

$$\partial_s(P_\infty Z)(s) - \delta M = G(s),$$

where $|G(s)| \leq C_1 g_1(s)$. By direct integration we obtain

$$\begin{aligned} |(P_\infty Z)(s) - \delta M s - c_0| &= \Big|\int_{\frac{s_0+s_1}{2}}^s G(\sigma)\,d\sigma\Big| \\ &\leq C_2\Big|\int_{\frac{s_0+s_1}{2}}^s \sqrt{\cosh(2\rho(s - \frac{s_0+s_1}{2}))}\,d\sigma\Big| \\ &\leq C_2\frac{\sqrt{2}}{\rho}\Big|\sinh(\rho(s - \frac{s_0+s_1}{2}))\Big| \\ &\leq C_2\frac{\sqrt{2}}{\rho}\cosh(\rho(s - \frac{s_0+s_1}{2})). \end{aligned}$$

Here $C_2 = C_1\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)/\sqrt{\cosh(\rho(s_1 - s_0))}$ and we have used the inequality $\sqrt{\cosh x} \leq \sqrt{2}\cosh(x/2)$. Therefore, there exists a uniquely

determined $\theta_0 \in S^1$ such that

$$\begin{aligned} &|(P_\infty Z)(s) - \varphi_s^{\delta f_\gamma}(\theta_0)| \\ \le\ & \frac{C_1\sqrt{2}}{\rho}\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\frac{\cosh(\rho(s-\frac{s_0+s_1}{2}))}{\sqrt{\cosh(\rho(s_1-s_0))}} \\ \le\ & \frac{C_1\sqrt{2}}{\rho}\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\frac{\cosh(\rho(s-\frac{s_0+s_1}{2}))}{\cosh(\rho(s_1-s_0)/2)}. \end{aligned}$$

The last inequality follows from $\cosh(x/2) \le \sqrt{\cosh x}$. A similar manipulation on (98) gives

$$|Q_\infty Z(s,\theta)| \le C\sqrt{2}\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\frac{\cosh(\rho(s-\frac{s_0+s_1}{2}))}{\cosh(\rho(s_1-s_0)/2)}.$$

The last two inequalities imply the conclusion of the Proposition in the case when $P_\infty Z(s)$, $s \in I_\delta$ stays away from $\mathrm{Crit}(f_\gamma)$.

If $P_\infty Z(s)$ is allowed to approach one of the critical points of f_γ, the estimate on $|Q_\infty Z(s)|$ stays the same, but the estimate involving $P_\infty Z(s)$ has to be modified as follows. In a suitable Morse coordinate chart around the critical point we can assume that $\nabla f_\gamma(x) = \pm Mx$, $M > 0$ and we have to study the equation

$$\partial_s(P_\infty Z)(s) \mp \delta M P_\infty Z(s) = G(s),$$

with $|G(s)| \le C_1 g_1(s)$. As in Proposition A.2 we have $P_\infty Z(s) = c(s)e^{\pm\delta Ms}$ with $e^{\pm\delta Ms}\partial_s c(s) = G(s)$. Then $c(s) = c_0 + \int_{\frac{s_0+s_1}{2}}^s e^{\mp\delta M\sigma}G(\sigma)\,d\sigma$ and there exists a $\theta_0 \in S^1$ such that $\varphi_s^{\delta f_\gamma}(\theta_0) = c_0 e^{\pm\delta Ms}$. We obtain

$$\begin{aligned} |(P_\infty Z)(s) - \varphi_s^{\delta f_\gamma}(\theta_0)| &\le \Big|\int_{\frac{s_0+s_1}{2}}^s e^{\pm\delta M(s-\sigma)}G(\sigma)\,d\sigma\Big| \\ &\le e^{\delta M(s_1-s_0)}\Big|\int_{\frac{s_0+s_1}{2}}^s G(\sigma)\,d\sigma\Big|. \end{aligned}$$

The last integral is bounded by

$$\frac{C_1\sqrt{2}}{\rho}\max(\|Q_\infty Z(s_0)\|, \|Q_\infty Z(s_1)\|)\frac{\cosh(\rho(s-\frac{s_0+s_1}{2}))}{\cosh(\rho(s_1-s_0)/2)}$$

as in the previous case and the conclusion follows. □

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